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Equilibrium Properties of Classical Systems with Long-Range Forces. BBGKY Equation, Neutrality, Screening, and Sum Rules

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We introduce a generalization of the BBGKY equation to define the equilibrium states for systems with long-range forces and study the properties of such states. We show that there are properties typical of short-range forces (shape independence, normal fluctuations, asymptotic behavior of correlation functions) and others which are typical of long-range forces (possible shape dependence, neutrality, sum rules and screening, abnormal fluctuations, boundedness of the internal electric field). If the force decreases at infinity faster than the Coulomb force, the properties will be those typical of short-range forces; on the other hand, if the force decreases at infinity as the Coulomb force or slower, the properties will be those typical of long-range forces.

KEY WORDS: BBGKY hierarchy; long-range force; Coulomb and jellium systems; neutrality; screening; canonical sum rules; charge and particle fluctuations.

1. INTRODUCTION

In this paper, we study the BBGKY hierarchy defining equilibrium states for infinite, continuous, classical systems with forces of arbitrary range. We show that there are some properties which are typical of systems with short-range forces and others which are typical of long-range forces. More precisely, if the forces decrease at infinity faster than the Coulomb force, the properties will be those of short-range forces, while if the forces decrease like the Coulomb force or slower, the properties will be those of long-range forces.

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For classical systems with short-range forces the usual equilibrium equations (KMS,[1] BBGKY,[2] KS,[3] DLR[4]) involve the fact that the force, or the potential, exerted on a particle by the rest of the system is finite. It is clear that for systems with long-range forces this property is far from being obvious; for example, for the one-dimensional "jellium" with background density ρ_B the first equation of the BBGKY hierarchy is formally given by

$$\frac{d}{dx}\rho^{(1)}(x) = -\beta\rho_B\rho^{(1)}(x)\int dy\,\mathrm{sign}(x-y) + \beta\int dy\,\mathrm{sign}(x-y)\rho^{(2)}(x,y)$$

Obviously the integrals do not have a well-defined meaning and one needs a generalization of the equilibrium equation in order to treat properly such non-absolutely convergent integrals.

The generalization consists in assuming that the BBGKY hierarchy is asymptotically satisfied when the effects of the force are first restricted to a sequence of finite regions (V_λ) converging to the whole space as $\lambda \to \infty$ (see Section 2 for a precise formulation). This amounts simply to considering the integrals occurring in the BBGKY hierarchy as limits of appropriate sequences of definite integrals. Clearly if the force is integrable, our prescription reduces to the usual BBGKY hierarchy. On the other hand, when the force is not integrable, a state verifying the generalized BBGKY hierarchy may depend genuinely on the sequence (V_λ); in particular, we expect that different sequences (V_λ) will distinguish different periodic structures.

Therefore one has to consider the sequence of regions (V_λ) as an additional parameter which, together with the usual thermodynamical parameters such as the temperature and the density, is necessary to characterize an equilibrium state.

It was shown in Ref. 5 that this generalized BBGKY hierarchy is equivalent to a generalization of the classical KMS condition and that both equations are satisfied by certain states of the one-dimensional Coulomb gas (the two-component system and the jellium).

We emphasize again that the prescription involving the sequence of volumes (V_λ) in the generalized BBGKY hierarchy, or in the KMS condition, applies to the correlation functions of the infinite system and must not be confused with the sequence of volumes used in the construction of such states using the thermodynamic limit of finite Gibbs states. We should recall that the equilibrium states of classical Coulomb systems have also been studied by means of thermodynamic limits of finite-volume Gibbs states (see Refs. 5 and 6 for the one-dimensional Coulomb gas; Refs. 5 and 7 for the one-dimensional jellium; and Ref. 8 for Coulomb systems in higher dimensions); however, the structure and consequences of the equilibrium equations obeyed by such states have not been investigated.

In this paper we shall study the general properties of equilibrium states defined by means of the generalized BBGKY hierarchy, but we shall not consider the relation between the two procedures mentioned above.

In Section 2, we introduce the systems of interest and formulate the generalized BBGKY hierarchy in terms of convenient physical parameters. To obtain such parameters we notice that a well-defined physical quantity is the *effective field* $\mathbf{E}(\mathbf{x})$ inside the system, which represents the combined effect of the field due to the particles together with the field due to possible outside sources (such as charges at infinity). Clearly a periodic, ionic Coulomb state will carry a nonzero electric field with the same periodicity. Translation-invariant states with nonzero electric field can also be exhibited in one dimension.[9] This effective field is a function of the state and should not be confused with the purely external field: it is a typical feature of systems with long-range forces that it is not possible to distinguish in this effective field the contribution due to the external sources as would be the case for systems with short-range forces. It thus follows that the external field is not a convenient parameter to label equilibrium states for systems with long-range forces and it should be replaced by the value $\mathbf{E}$ of this effective field at a given point, say the origin. We thus parametrize the equilibrium states by the usual thermodynamical parameters (T, ρ,...) together with the effective field at the origin $\mathbf{E} = \mathbf{E}(0)$ and the sequence of regions (V_λ); we then call "regular equilibrium states," the states parametrized in this manner which are solutions of the BBGKY hierarchy and satisfy certain smoothness and clustering conditions.

Section 3 is devoted to the study of the general properties of regular equilibrium states. We first investigate to what extent a regular equilibrium state will depend on the sequence of regions (V_λ). The main result is that the role played by (V_λ) is related to the range of the force: for forces decreasing at infinity faster than the Coulomb force regular equilibrium states do not depend on the geometry of the (V_λ), whereas for the Coulomb force (or forces with slower decrease), sequences (V_λ) having asymptotically different shapes may distinguish different equilibrium states. We then discuss the transformation properties of the state under various symmetry operations, and we deduce the consequences on the effective field of the invariance of the state under translations, rotations, charge conjugation, and scaling.

In Section 4, we show that it follows directly from the equilibrium equation that a regular equilibrium state invariant under a discrete subgroup of the translations (Bravais lattice) must be locally neutral as soon as the force decreases like the Coulomb force or slower.

On the other hand, for forces with faster decrease at infinity, nothing can be concluded about neutrality. This result also shows that for one-component systems (without a rigid background) there does not exist regular equilibrium states if the force decreases like the Coulomb force or slower. Therefore, the analysis of this section shows once more that the Coulomb force is precisely the borderline where new properties appear.

In Section 5, we deduce another essential characteristic of systems with long-range forces: the canonical sum rules. The sum rules are a hierarchy of integral relations that link the n-point to the $(n + 1)$-point correlation function. We show that these sum rules follow necessarily from the equilibrium equation as soon as the rate of decrease of the truncated correlation functions (i.e., the clustering) is faster than the decrease of the force at infinity. The latter condition is essentially verified when the state has the rather mild property of $\mathscr{L}^1$-clustering and thus any regular equilibrium state of a system with long-range force (in particular, Coulomb systems) will obey the hierarchy of sum rules in addition to the BBGKY hierarchy. These sum rules are identical to those which exist by the very definition of the correlation functions in a finite canonical ensemble of several kinds of particles. Although these constraints are trivially satisfied for such finite systems, they are obviously not true for an infinite system of free particles and they are not expected to remain true after the thermodynamic limit in the case of short-range forces; in other words, the validity of the sum rules must be seen as a feature specific to systems with long-range forces.

We develop in the last section some consequences of the sum rules; the most striking is that the mean square fluctuations of the charge in a region Λ are not extensive with Λ. This shows that the charge does not have the usual behavior of macroscopic observables; further aspects of the charge fluctuations are studied in Ref. 10. This result has several interesting implications: since in a one-component Coulomb system (jellium) the charge is identical with the particle number, we conclude that the fluctuations of the particle number are not extensive in the jellium. This implies in turn that the jellium has always zero compressibility. Conversely, consider a compressible fluid constituted by a single kind of particles interacting with a short-range force. Since the compressibility is nonzero, the sum rule cannot hold true and therefore we must conclude that the decrease of the truncated functions cannot be faster than the decrease of the force. Thus we get a lower bound on the clustering. With an additional assumption on the derivatives of the truncated functions, we can improve this lower bound and show that the truncated correlation functions cannot decrease faster than the potential itself. It thus follows from a result of Groeneveld[11] that the two-point truncated correlation function decreases exactly like the potential at infinity at low density.

Finally, the sum rules provide an upper bound on the possible value of the effective field in translation-invariant Coulomb systems with several components.

2. THE SYSTEM AND THE BBGKY HIERARCHY

We consider a system of N different types of "charged" particles in $\mathbb{R}^\nu$, with "charges" σ in Σ, where Σ is an N-point subset of $\mathbb{R}\backslash\{0\}$. We denote the coordinate and charge of a particle by

$$q = (x\sigma), \qquad x = \{x_\alpha; \alpha = 1,..., \nu\} \in \mathbb{R}^\nu, \quad \sigma \in \Sigma \tag{1}$$

and for any $V \subset \mathbb{R}^\nu$ we write $\int_V dq = \int_V dx \sum_{\sigma\in\Sigma}$.

The particles interact by means of a two-body force $F(q_1, q_2)$ of the form

$$F(q_1, q_2) = \sigma_1\sigma_2 F(x_1 - x_2) \tag{2}$$

where $F(x) = \{F_\alpha(x); \alpha = 1,..., \nu\}$ is independent of the charge.

Moreover, the particles are imbedded in the uniform background of fixed, negative unit charges with particle density $\rho_B \geqslant 0$. This uniform background acts on the particles as a one-body force formally given by

$$F_B(q) = -\rho_B\sigma \int F(x - y)\, dy \tag{3}$$

Finally the particles are subjected to the effects of a constant force

$$F^{(1)}(q) = \sigma D \tag{4}$$

where $D = \{D_\alpha; \alpha = 1,..., \nu\}$ is a fixed vector in $\mathbb{R}^\nu$.

Systems of particular interest are *Coulomb systems* with the two-body force given by

$$F(x) = x/|x|^\nu \tag{5}$$

or by a smeared Coulomb force

$$F(x) = \int g(y) \frac{x - y}{|x - y|^\nu}\, dy \tag{6}$$

where $g(y)$ is a smooth and rapidly decreasing function.

The following special cases of Coulomb systems are often studied: (a) the *one-component plasma* or *jellium* with $\rho_B \neq 0$, $N = 1$, and $\sigma = +1$; (b) the *two-component plasma* with $\rho_B = 0$, $N = 2$, and $\sigma_1 = -\sigma_2$.

We have introduced the constant force (4) in order to discuss later Coulomb systems with nonvanishing constant external electric fields.

We shall consider a general class of forces satisfying the following conditions.

Condition (F1) on the force. (i) $F_\alpha(x)$ is locally integrable and continuous in any open set which does not contain the origin $x = 0$. (ii) $F_\alpha(x)$ is continuously differentiable and bounded for $|x| > r$.

In the following we shall discuss states which are described by correlation functions[2] $\rho^{(n)}(q_1,..., q_n) = \rho^{(n)}(x_1\sigma_1, x_2\sigma_2,..., x_n\sigma_n)$, $n = 1, 2,...$. The correlation functions are positive, symmetric under permutations of the arguments $q_1,..., q_n$, and satisfy the following condition:

Condition (S1) on the states. (i) The $\rho^{(n)}(q_1,..., q_n)$, as functions of $x_1,..., x_n$, are of class C^0 everywhere in $\mathbb{R}^{\nu n}$, and of class C^1 for any open set which does not contain coincident particles. (ii) The $\rho^{(n)}(q_1,..., q_n)$ are uniformly bounded in $\mathbb{R}^{\nu n}$.

We then define the *equilibrium states* by means of the solutions of the following BBGKY equation: Let (V_λ) be a sequence of bounded space regions converging to $\mathbb{R}^\nu$ as $\lambda \to \infty$. We say that *the state ρ with correlations $\rho^{(n)}(q_1,..., q_n)$ is an equilibrium state with respect to the parameter D and the sequence of volumes* (V_λ) [for temperature β^{-1}, background density ρ_B, and two-body force $F(x)$] if ($\nabla_1 = \{\partial/\partial x_{1,\alpha};\ \alpha = 1,..., \nu\}$)

$$\begin{aligned}\nabla_1\rho^{(n)}(q_1,..., q_n) = \beta \lim_{\lambda\to\infty} \Bigg\{ \Bigg[\sigma_1 D - \rho_B\sigma_1 \int_{V_\lambda} F(x_1 - y)\, dy \\ + \sum_{j=2}^{n} F(q_1, q_j) \Bigg] \rho^{(n)}(q_1,..., q_n) \\ + \int_{V_\lambda} dq\, F(q_1, q)\rho^{(n+1)}(q_1,..., q_n, q) \Bigg\} \end{aligned} \tag{7}$$

This definition deserves the following comments:

(i) The hierarchy of correlation functions is understood to describe an equilibrium state of an *infinitely extended system*. The sequence (V_λ) occurring in (7) applies to the state of the infinite system and must not be confused with sequences of volumes used in the construction of such states by taking the thermodynamic limit of finite Gibbs states.

(ii) If the force is integrable in the whole space, it follows from the general conditions F1 and S1 that the limit in (7) exists and is independent of the sequence (V_λ). In this case (7) reduces to the usual BBGKY hierarchy and ρ is independent of (V_λ).

(iii) If the force is not integrable, a limiting procedure is needed to sum the forces up to infinity and *the state ρ may depend genuinely on the sequence* (V_λ): different sequences can distinguish different physical states (for instance, different periodic structures).

(iv) We adopt (7) as a definition of equilibrium states for systems with long-range interactions, and we want to discuss the properties of the solutions of (7) assuming that they exist. Some motivations can be found in Ref. 5, where it is shown that the BBGKY system in the form (7) is equivalent to a natural generalization of the classical KMS equilibrium condition. Moreover, it is shown in Ref. 5 that the simplest cases of Coulomb systems (the one-

[2] We shall also use the notation $\rho^{(n)}(q_1,..., q_n) = \rho^{(n)}_{\sigma_1...\sigma_n}(x_1,..., x_n)$.

dimensional one- and two-component plasmas) do satisfy Eq. (7) for appropriate sequences (V_λ).

Since we are mainly interested in the study of extremal states, we shall restrict our investigations to equilibrium states which satisfy the following *C1 clustering* condition:

$$\text{(C1)} \qquad \int_{\mathbb{R}^\nu} |\rho_T^{(n)}(q_1,..., q_n)| \, dq_1 < \infty, \qquad n \geqslant 2$$

for all $q_2,..., q_n$, where the $\rho_T^{(n)}(q_1,..., q_n)$ are the truncated correlation functions defined in the usual way.

To study the solutions of Eq. (7), we would like first to replace the limit of the sum of integrals in Eq. (7) by a simpler limit involving only one integral. To do so, it is convenient to introduce the *charge density* c_ρ associated with the state ρ

$$c_\rho(x) = \sum_{\sigma \varepsilon \Sigma} \sigma \rho^{(1)}(x\sigma) - \rho_B \tag{8}$$

The main observation is that if a state is C1-clustering, the limit on the rhs of Eq. (7) implies the existence of the following limit:

$$\lim_{\lambda \to \infty} \int_{V_\lambda} F(x - y) c_\rho(y) \, dy$$

which involves only the one-point correlation function.

Proposition 1. For any C1-clustering equilibrium state ρ,

$$\text{(i)} \qquad \lim_{\lambda \to \infty} \int_{V_\lambda} F(x - y) c_\rho(y) \, dy \tag{9}$$

exists except possibly at these points x where $\rho^{(n)}(x\sigma_1, q_2,..., q_n) = 0$ for all values of $n, \sigma_1, q_2,..., q_n$.

(ii) Equation (7) is equivalent to the following equation:

$$\begin{aligned}\nabla_1 \rho^{(n)}(q_1,..., q_n) = \beta \Bigg\{ \sigma_1 D &+ \sigma_1 \left[\lim_{\lambda \to \infty} \int_{V_\lambda} F(x_1 - y) c_\rho(y) \, dy \right] \\ &+ \sum_{j=2}^{n} F(q_1, q_j) \Bigg\} \rho^{(n)}(q_1,..., q_n) \\ &+ \beta \int_{\mathbb{R}^\nu} dq \, F(q_1, q)[\rho^{(n+1)}(q_1,..., q_n, q) - \rho^{(n)}(q_1,..., q_n)\rho^{(1)}(q)]\end{aligned} \tag{10}$$

where we can replace $[\lim_{\lambda \to \infty} \int_{V_\lambda} F(x_1 - y) c_\rho(y) \, dy]$ by zero whenever the limit does not exist.

Proof. If the state satisfies the clustering condition C1, we notice that the function

$$F_\alpha(q_1, q)[\rho^{(n+1)}(q_1,\dots, q_n, q) - \rho^{(n)}(q_1,\dots, q_n)\rho^{(1)}(q)]$$

is integrable in q. Indeed this function is integrable locally by the assumptions F1 and S1, and integrable at infinity by the clustering and the fact that $F(q_1, q)$ remains bounded. Thus

$$\begin{aligned}\lim_{\lambda\to\infty} \int_{V_\lambda} dq\, F_\alpha(q_1, q)[\rho^{(n+1)}(q_1,\dots, q_n, q) - \rho^{(n)}(q_1,\dots, q_n)\rho^{(1)}(q)] \\ = \int_{\mathbb{R}^\nu} dq\, F_\alpha(q_1, q)[\rho^{(n+1)}(q_1,\dots, q_n, q) - \rho^{(n)}(q_1,\dots, q_n)\rho^{(1)}(q)] \quad (11)\end{aligned}$$

exists and is independent of the sequence (V_λ). Subtracting and adding the function $\int_{V_\lambda} dq\, F_\alpha(q_1, q)\rho^{(n)}(q_1,\dots, q_n)\rho^{(1)}(q)$ to the integrand of the last term of (7), we see that (7) and (11) imply that the limit

$$\lim_{\lambda\to\infty} \left[\int_{V_\lambda} F_\alpha(x_1 - y)c_\rho(y)\, dy\, \rho^{(n)}(q_1,\dots, q_n)\right] \tag{12}$$

exists for all $n, q_1,\dots, q_n$ and (7) implies (10). Furthermore, (12) implies the existence of the limit (9) for all points x such that $\rho^{(n)}(x\sigma_1, q_2,\dots, q_n) \neq 0$ for some $n, \sigma_1, q_2,\dots, q_n$ and otherwise the limit (12) is zero. ■

Remarks

1. If the equilibrium state is C1-clustering and invariant under translations, then $\rho^{(1)}(x\sigma) = \rho_\sigma \neq 0$ implies that the limit (9) exists for all x.

In any case this limit exists at each point x where $\rho^{(1)}(x\sigma) \neq 0$.

2. One should note the following important feature, which is specific to systems with long-range forces. If ρ is a C1-clustering equilibrium state with respect to D and (V_λ), then for any $a \in \mathbb{R}^\nu$ the *same state* is obviously an equilibrium state with respect to the *new sequence of volumes* $(V_\lambda + a)$, $V_\lambda + a = \{x + a; x \in V_\lambda\}$, and the *new parameter* $D(a)$,[3]

$$D(a) = D + \lim_{\lambda\to\infty} \left\{\int_{V_\lambda} F(x - y)c_\rho(y)\, dy - \int_{V_\lambda + a} F(x - y)c_\rho(y)\, dy\right\} \tag{13}$$

If the force has long range, the limit in (13) can be different from zero (see, for instance, the case of the one-dimensional jellium, Section 3.3).

This shows (as already discussed in Ref. 5) that D does not have a well-defined meaning without the specification of the sequence of volumes (V_λ). In particular, D cannot represent a constant external applied field.

On the other hand, the *new state* $\tau_a\rho$ obtained by translation of ρ is clearly an equilibrium state with respect to the *same parameter* D and the *new*

[3] We shall see that for a large class of long-range forces, including Coulomb, $D(a)$ is independent of x (see Proposition 2 below).

sequence of volumes $(V_\lambda + a)$. Therefore if the state is not $\mathbb{R}^\nu$-invariant, the parameter D does not distinguish between the state and its translates.

This discussion leads us to the following conclusion: The parameters D and (V_λ) are entangled and D does not provide a physical nor a convenient labeling of equilibrium states.

For these reasons we want to label the equilibrium states by means of a new parameter E with the property that if ρ is an equilibrium state with respect to E and (V_λ), then it is also an equilibrium state with respect to E and $(V_\lambda + a)$; in this manner the parameters E and (V_λ) are partially disentangled and it will be possible to give a physical interpretation of E.

To find such a parameter we notice that although the quantities D and $\lim_{\lambda\to\infty} \int_{V_\lambda} F(x-y)c_\rho(y)\,dy$ do not have a direct physical interpretation if the limit (13) is nonzero, their sum, which we denote

$$E_\rho(x) = D + \lim_{\lambda\to\infty} \int_{V_\lambda} F(x-y)c_\rho(y)\,dy \tag{14}$$

will have a physical meaning; indeed it follows from (10) that

$$\sigma_1 E_\rho(x)\rho^{(1)}(q) = \frac{1}{\beta}\left\{\nabla\rho^{(1)}(q) - \int_{\mathbb{R}^\nu} d\bar{q}\, F(q,\bar{q})\rho_T^{(2)}(q,\bar{q})\right\}$$

and thus $E_\rho(x)$ depends only on the state and not on the labelling used.

We shall then call $E_\rho(x)$ the *effective field at the point x in the state ρ.*

The main idea is to label an equilibrium state by E, (V_λ) where E is the value of its effective field at the origin,[4] i.e.,

$$E = E_\rho(0) = D + \lim_{\lambda\to\infty} \int_{V_\lambda} F(-y)c_\rho(y)\,dy \tag{15}$$

We note that if ρ is an equilibrium state with respect to D, (V_λ), then its translate $\tau_a\rho$ is an equilibrium state with respect to D, $(V_\lambda + a)$ and thus the effective field in the state $\tau_a\rho$ is given by

$$\begin{aligned} E_{\tau_a\rho}(x) &= D + \lim_{\lambda\to\infty} \int_{V_\lambda + a} F(x-y)c_{\tau_a\rho}(y)\,dy \\ &= D + \lim_{\lambda\to\infty} \int_{V_\lambda} F(x-a-y)c_\rho(y)\,dy \end{aligned}$$

i.e.

$$E_{\tau_a\rho}(x) = E_\rho(x-a) \tag{16}$$

In the rest of this paper we shall then discuss the properties of equilibrium states in terms of E instead of D.

We remark first that it follows from (16) that if ρ is an equilibrium state

[4] Without loss of generality we can assume that $\rho^{(1)}(x\sigma)$ is not zero at the origin and thus we know by Proposition 1 that the limit (15) exists.

with respect to E, (V_λ), then $\tau_a\rho$ is an equilibrium state with respect to E_a, $(V_\lambda + a)$, where

$$E_a = E_{\tau_a\rho}(0) = E_\rho(-a) \tag{17}$$

Furthermore, in terms of the parameters E, (V_λ), the effective field is given by

$$E_\rho(x) = E + \lim_{\lambda\to\infty} \int_{V_\lambda} [F(x-y) - F(-y)]c_\rho(y)\,dy \tag{18}$$

and the BBGKY equation becomes

$$\begin{aligned}
&\nabla_1 \rho^{(n)}(q_1,\dots,q_n)\\
&\quad = \beta\Bigg(\sigma_1\Big\{E + \lim_{\lambda\to\infty}\int_{V_\lambda} [F(x_1-y) - F(-y)]c_\rho(y)\,dy\Big\}\\
&\qquad + \sum_{j=2}^{n} F(q_1, q_j)\Bigg)\rho^{(n)}(q_1,\dots,q_n)\\
&\qquad + \beta\int_{\mathbb{R}^\nu} dq\, F(q_1, q)[\rho^{(n+1)}(q_1,\dots,q_n,q) - \rho^{(n)}(q_1,\dots,q_n)\rho^{(1)}(q)]
\end{aligned} \tag{19}$$

Clearly any equilibrium state defined by a solution of Eq. (10) also yields a solution of Eq. (19). Conversely any solution ρ of Eq. (19) such that

$$\lim_{\lambda\to\infty}\int_{V_\lambda} F(x-y)c_\rho(y)\,dy$$

exists yields a solution of Eq. (10).

We shall then adopt Eq. (19) as the definition of equilibrium states and discuss the properties of the solutions of this equation.

3. GENERAL PROPERTIES OF REGULAR EQUILIBRIUM (RE) STATES

We call *regular equilibrium state* (RE state) a state which is C1-clustering, satisfies S1, and is a solution of the BBGKY hierarchy (19).

In this section we shall first study the dependence of a RE state $\rho_{E,(V_\lambda)}$ on the sequence (V_λ); we shall then give the transformation properties of $\rho_{E,(V_\lambda)}$ under various symmetry operations, and finally we shall discuss the physical interpretation of the parameter E.

3.1. Dependence of $\rho_{E,(V_\lambda)}$ on (V_λ)

To discuss to what extent $\rho_{E,(V_\lambda)}$ depends on the sequence (V_λ) we first have to specify how (V_λ) tends to $\mathbb{R}^\nu$.

In the following we consider sequences (V_λ) converging to $\mathbb{R}^\nu$ in the following manner:

(a) $V_\lambda \to \mathbb{R}^\nu$ in the sense of van Hove, i.e., if V_λ^h is the set of points that are at a distance less than or equal to h from the boundary ∂V_λ of V_λ, then $|V_\lambda^h|/|V_\lambda| \to 0$ as $\lambda \to \infty$.

(b) For each V_λ, there exist two balls B_λ^- and B_λ^+ centered at the origin with radii R_λ^- and R_λ^+ such that $B_\lambda^- \subseteq V_\lambda \subseteq B_\lambda^+$ and $R_\lambda^- \to \infty$ as $\lambda \to \infty$, with $R_\lambda^+ = O(R_\lambda^-)$.

Furthermore, in some cases, we shall also need the following condition:

(c) $|V_\lambda^h|/|V_\lambda| = O(1/R_\lambda^-)$ as $\lambda \to \infty$.

Condition (b) means roughly that "V_λ extends at the same speed in all directions" as $\lambda \to \infty$.

Condition (c) means that the volume of a boundary layer of fixed thickness is of the order $(R_\lambda^-)^{\nu-1}$ as $\lambda \to \infty$.

We shall now show that for a given two-body force the limit

$$\lim_{\lambda\to\infty} \int_{V_\lambda} [F(x-y) - F(-y)]c_\rho(y)\, dy$$

is independent of (V_λ) within a certain class of sequences (V_λ) which we shall specify. *This class will depend on the range of the force.* We shall thus conclude that if $\rho_{E,(V_\lambda)}$ is a RE state with respect to E, (V_λ), then $\rho_{E,(V_\lambda)}$ is also a RE state with respect to E and (V_λ') whenever (V_λ') is in the same class as (V_λ).

Let us then introduce the following relations between sequences (V_λ):

1. $(V_\lambda^1)\, \alpha_1\, (V_\lambda^2)$ if $V_\lambda^1 \triangle V_\lambda^2 = O(V_\lambda^1 \cap V_\lambda^2)$ as $\lambda \to \infty$.
2. $(V_\lambda^1)\, \alpha_2\, (V_\lambda^2)$ if $V_\lambda^1 \triangle V_\lambda^2 = o(V_\lambda^1 \cap V_\lambda^2)$ as $\lambda \to \infty$.

   [Here $V_\lambda^1 \triangle V_\lambda^2$ is the symmetric difference of the regions V_λ^1 and V_λ^2, i.e., $V_\lambda^1 \triangle V_\lambda^2 = (V_\lambda^1 \cup V_\lambda^2)\backslash(V_\lambda^1 \cap V_\lambda^2)$.]
3. $(V_\lambda^1)\, \alpha_3\, (V_\lambda^2)$ if V_λ^2 is some translate of the region V_λ^1, i.e., $V_\lambda^2 = \{x + a; x \in V_\lambda^1\} = V_\lambda^1 + a$.

Lemma 1. Consider sequences (V_λ) converging to $\mathbb{R}^\nu$ in the senses (a) and (b). Then the relations $\alpha_1, \alpha_2, \alpha_3$ are equivalence relations and $\alpha_3 \Rightarrow \alpha_2 \Rightarrow \alpha_1$.

The proof of the lemma is given in the Appendix.

The property that distinguishes these different equivalence relations is that in cases α_2 and α_3 the volume of the symmetric difference between the regions V_λ^1 and V_λ^2 is not allowed to grow extensively as $\lambda \to \infty$. If $(V_\lambda^1)\, \alpha_1\, (V_\lambda^2)$, then V_λ^1 and V_λ^2 may have asymptotically completely different shapes, whereas $(V_\lambda^1)\, \alpha_2\, (V_\lambda^2)$ means roughly that V_λ^1 and V_λ^2 must have asymptotically a similar shape. When $(V_\lambda^1)\, \alpha_3\, (V_\lambda^2)$, then V_λ^1 and V_λ^2 have obviously the same shape.

Proposition 2. Let $\rho_{E,(V_\lambda)}$ be a RE state.

(i) If $F(x) \in \mathscr{L}^1(\mathbb{R}^\nu)$, then $\rho_{E,(V_\lambda')} = \rho_{E,(V_\lambda)}$ for any sequence (V_λ').

Assume that the sequence (V_λ) has the properties (a) and (b).

(ii) If $|x|^\nu|\nabla F_\alpha(x)| = o(1)$, then $\rho_{E,(V_\lambda')} = \rho_{E,(V_\lambda)}$ for any sequence (V_λ') such that $(V_\lambda')\ \alpha_1\ (V_\lambda)$.

(iii) If $|x|^\nu|\nabla F_\alpha(x)| = O(1)$, then $\rho_{E,(V_\lambda')} = \rho_{E,(V_\lambda)}$ for any sequence (V_λ') such that $(V_\lambda')\ \alpha_2\ (V_\lambda)$.

Assume, moreover, that the sequence (V_λ) has the property (c).

(iv) If $|x|^{\nu-1}|\nabla F_\alpha(x)| = o(1)$, then $\rho_{E,(V_\lambda')} = \rho_{E,(V_\lambda)}$ for any sequence (V_λ') such that $(V_\lambda')\ \alpha_3\ (V_\lambda)$, i.e., $\rho_{E,(V_\lambda)} = \rho_{E,(V_\lambda+a)}\ \forall a \in \mathbb{R}^\nu$.

Proof. We show in all cases that the limit of

$$\int_{V_\lambda} [F_\alpha(x-y) - F_\alpha(-y)]c_\rho(y)\,dy - \int_{V_\lambda'} [F_\alpha(x-y) - F_\alpha(-y)]c_\rho(y)\,dy \tag{20}$$

as $\lambda \to \infty$ is zero. This implies that the limit with the sequence (V_λ') exists and is equal to the limit with the sequence (V_λ).

The case (i) is obvious since the charge density is supposed to be uniformly bounded.

For the other cases, the quantity (20) is majorized by

$$\|c_\rho\|_\infty \int_{V_\lambda \Delta V_\lambda'} |F_\alpha(x-y) - F_\alpha(-y)|\,dy$$

Let $\tilde{R}_\lambda = \min(R_\lambda^-, R_\lambda^{-\prime})$, with R_λ^- and $R_\lambda^{-\prime}$ the radii of the internal spheres B_λ^- and $B_\lambda^{-\prime}$, and choose λ such that $\tilde{R}_\lambda > r + |x|$ (r given in F1). If $y \in V_\lambda \triangle V_\lambda'$, we have clearly

$$|y - x| \geqslant \big|\,|y| - |x|\,\big| \geqslant \tilde{R}_\lambda - |x| > r \tag{21}$$

Under the assumptions made on the force in F1 and (ii) or (iii), we get for $y \in V_\lambda \triangle V_\lambda'$, using the limited Taylor expansion theorem,

$$\begin{aligned} |F_\alpha(x-y) - F_\alpha(-y)| &= |x\,\nabla F_\alpha(-y+\theta x)|, \qquad 0 \leqslant \theta \leqslant 1 \\ &\leqslant \nu|x| \sup_{y \in V_\lambda \Delta V_\lambda'} \frac{A}{|y - \theta x|^\nu} \leqslant \nu|x| \frac{A}{(\tilde{R}_\lambda - |x|)^\nu} \\ &\leqslant \mathrm{cst}\,\frac{A}{(\tilde{R}_\lambda)^\nu} \end{aligned} \tag{22}$$

where the constant A can be chosen arbitrarily small in the case (ii). Thus

$$\int_{V_\lambda \Delta V_\lambda'} |F_\alpha(x-y) - F_\alpha(-y)|\,dy \leqslant \mathrm{cst}\,A\,\frac{|V_\lambda \triangle V_\lambda'|}{(\tilde{R}_\lambda)^\nu} \leqslant \mathrm{cst}\,A\,\frac{|V_\lambda \triangle V_\lambda'|}{|V_\lambda \cap V_\lambda'|} \tag{23}$$

The last inequality follows from the property (b) of our sequences of volumes (V_λ):

$$\frac{|V_\lambda \triangle V_\lambda'|}{(\tilde{R}_\lambda)^\nu} \leqslant \left[\max\left(\frac{R_\lambda^+}{R_\lambda^-}, \frac{R_\lambda^{+\prime}}{R_\lambda^{-\prime}}\right)\right]^\nu \frac{|V_\lambda \triangle V_\lambda'|}{[\min(R_\lambda^+, R_\lambda^{+\prime})]^\nu} \leqslant \mathrm{cst}\, \frac{|V_\lambda \triangle V_\lambda'|}{|V_\lambda \cap V_\lambda'|}$$

In the case (ii), $|V_\lambda \triangle V_\lambda'|/|V_\lambda \cap V_\lambda'|$ remains bounded as $\lambda \to \infty$ and we can let $A \to 0$. In case (iii),

$$\lim_{\lambda \to \infty} (|V_\lambda \triangle V_\lambda'|/|V_\lambda \cap V_\lambda'|) = 0$$

For case (iv), $V_\lambda' = \{x + a; x \in V_\lambda\}$ for some a is simply a translate of V_λ. Without loss of generality we can assume that (V_λ) and (V_λ') have the property (b) with respect to the same sequences of balls (B_λ^-) and (B_λ^+) with radii R_λ^- and R_λ^+. Therefore we can replace $\tilde{R}_\lambda$ by R_λ^- in the estimates (22). The inequalities (22) and (23) become

$$|F_\alpha(x - y) - F_\alpha(-y)| \leqslant \nu|x| \frac{A}{(R_\lambda^-)^{\nu-1}} \tag{24}$$

and

$$\int_{V_\lambda \Delta V_\lambda'} |F_\alpha(x - y) - F_\alpha(-y)|\, dy \leqslant \mathrm{cst}\, A \frac{|V_\lambda \triangle V_\lambda'|}{(R_\lambda^-)^{\nu-1}}$$
$$\leqslant \mathrm{cst}\, A\, R_\lambda^- \frac{|V_\lambda \triangle V_\lambda'|}{|V_\lambda|} \tag{25}$$

where A is arbitrarily small.

Now in virtue of (c), we have

$$|V_\lambda \triangle V_\lambda'|/|V_\lambda| \leqslant 2|V^{|a|}|/|V_\lambda| \leqslant \mathrm{cst}/R_\lambda^- \tag{26}$$

where $V_\lambda^{|a|}$ is the set of points that are at a distance less than or equal to $|a|$ from the boundary of V_λ. The conjunction of (25) and (26) proves (iv). ■

In order to make the content of the proposition more explicit, we notice that if the force satisfies the conditions

$$\begin{aligned} |x|^\gamma |F_\alpha(x)| &= O(1), \qquad |x| \to \infty \\ |x|^{\gamma+1} |\nabla F_\alpha(x)| &= O(1), \qquad |x| \to \infty \end{aligned} \qquad (\gamma \geqslant 0)$$

(as will be the case in Sections 4 and 5), then the equivalence class of (V_λ) is specified by the asymptotic behavior of the force as given in Table I.

There are three important observations to be made at this point.

1. The RE states are parametrized by β, ρ_B, E, *and* the equivalence class of (V_λ), where the equivalence relation depends on the range of the force. However, if the two-body force is $\mathscr{L}^1$, the states are simply parametrized as usual by β, ρ_B, and E.

Table I

	Asymptotic behavior of the force	Equivalence class of sequences of volumes	
Case (i)	$\gamma > \nu$	All sequences are equivalent	V_λ^1 and V_λ^2 allowed to have completely different shapes
Case (ii)	$\nu \geq \gamma > \nu - 1$	$(V_\lambda^1)\ \alpha_1\ (V_\lambda^2)$	
Case (iii)	$\gamma = \nu - 1$	$(V_\lambda^1)\ \alpha_2\ (V_\lambda^2)$	V_λ^1 and V_λ^2 have asymptotically the same shape
Case (iv)	$\nu - 1 > \gamma > \nu - 2$	$(V_\lambda^1)\ \alpha_3\ (V_\lambda^2)$	

2. The Coulomb force ($\gamma = \nu - 1$) is just at the borderline between cases (ii) and (iv): if the force decreases faster than Coulomb, then case (ii) shows that the state will not depend on the shape of the regions (V_λ), whereas for the Coulomb force, case (iii) shows that different equivalence classes of regions (V_λ) with different asymptotic shapes may distinguish different states.

3. Two sequences (V_λ^1) and (V_λ^2) such that one is the translate of the other are equivalent as long as the force decreases faster than $|x|^{-\gamma}$ with $\gamma > \nu - 2$. This means that $\rho_{E,(V_\lambda)}$ may depend only on the asymptotic shape of the V_λ, but not on the absolute location of the V_λ in space.

In the next section, we shall consider special sequences obtained by dilatations of a fixed region V_0:

$$V_\lambda = \{\lambda x;\, x \in V_0\} = \lambda V_0, \qquad \lambda \in \mathbb{R}^+$$

where V_0 is assumed to have a boundary ∂V_0 which is a piecewise C^1-manifold in $\mathbb{R}^\nu$. The states obtained in this manner are then parametrized by E and V_0 and we introduce the notation

$$\rho_{E,V_0} = \rho_{E,(\lambda V_0)}$$

The sequences of dilated regions always have the properties (a), (b), and (c) above; furthermore, $(\lambda V_0^1)\ \alpha_1\ (\lambda V_0^2)$, but $(\lambda V_0^1)\ \alpha_2\ (\lambda V_0^2)$ if and only if $V_0^1 = V_0^2$.

If we can find $V_0' \neq V_0$ such that ρ_{E,V_0} is not an equilibrium state with respect to $(\lambda V_0')$, we say that ρ_{E,V_0} is shape dependent. Therefore we conclude from Proposition 2 that if the force decreases faster than Coulomb, any RE state is shape independent. However, if the force decreases asymptotically like the Coulomb force (or has a slower decrease), ρ_{E,V_0} may be genuinely shape dependent.

3.2. Transformation Properties of RE States

We examine the transformation properties of a RE state under various symmetry operations.

Let $\rho_{E,(V_\lambda)}$ be a RE state with correlations $\rho^{(n)}_{\sigma_1,\ldots,\sigma_n}(x_1,\ldots, x_n)$. With a a vector in $\mathbb{R}^\nu$, R a rotation, and I the space inversion, we consider the transformed states $\tau_a\rho_{E,(V_\lambda)}$, $\tau_R\rho_{E,(V_\lambda)}$, and $\tau_I\rho_{E,(V_\lambda)}$ defined respectively by the correlation functions $\rho^{(n)}_{\sigma_1,\ldots,\sigma_n}(x_1 - a,\ldots, x_n - a)$, $\rho^{(n)}_{\sigma_1,\ldots,\sigma_n}(R^{-1}x_1,\ldots, R^{-1}x_n)$, and $\rho^{(n)}_{\sigma_1,\ldots,\sigma_n}(-x_1,\ldots, -x_n)$. If both σ and $-\sigma$ belong to Σ, we also introduce the charge conjugate state $\tau_C\rho_{E,(V_\lambda)}$ defined by the correlation functions $\rho^{(n)}_{-\sigma_1,\ldots,-\sigma_n}(x_1,\ldots, x_n)$.

Proposition 3. Let $\rho_{E,(V_\lambda)}$ be a RE state with respect to ρ_B, E, (V_λ).

(i) The translated state $\tau_a\rho_{E,(V_\lambda)}$ is a RE state with respect to ρ_B, E_a, and $(V_\lambda + a)$, where

$$E_a = E + \lim_{\lambda\to\infty}\int_{V_\lambda}[F(-a - y) - F(-y)]c_\rho(y)\,dy \tag{27}$$

i.e.,

$$\tau_a\rho_{E,(V_\lambda)} = \rho_{E_a,(V_\lambda+a)}$$

Furthermore, the effective field (18) in the state $\tau_a\rho_{E,(V_\lambda)}$ is given by

$$E_{\tau_a\rho}(x) = E_\rho(x - a) \tag{28}$$

(ii) If the force is covariant under rotations, then $\tau_R\rho_{E,(V_\lambda)}$ is a RE state with respect to ρ_B, RE, and (RV_λ), i.e.,

$$\tau_R\rho_{E,(V_\lambda)} = \rho_{RE,R(V_\lambda)},\qquad RV_\lambda = \{Rx;\, x\in V_\lambda\}$$

(iii) If $F(x) = -F(-x)$, then $\tau_I\rho_{E,(V_\lambda)}$ is a RE with respect to ρ_B, $IE = -E$, and (IV_λ), i.e.,

$$\tau_I\rho_{E,(V_\lambda)} = \rho_{-E,(IV_\lambda)},\qquad IV_\lambda = \{x;\, -x\in V_\lambda\}$$

(iv) If $\Sigma = -\Sigma$, then $\tau_C\rho_{E,(V_\lambda)}$ is a RE state with respect to $-\rho_B$, $-E$, and (V_λ), i.e.,

$$\tau_C\rho_{\rho_B,E,(V_\lambda)} = \rho_{-\rho_B,-E,(V_\lambda)}$$

Proof. Denote by $\rho_a^{(n)}(q_1,\ldots, q_n)$ the correlation functions of the translated state $\tau_a\rho_{E,(V_\lambda)}$ and change x_j into $x_j - a$ in Eq. (19). Since the two-body force is itself translation invariant, all terms of Eq. (19) can be expressed with the $\rho_a^{(n)}(q_1,\ldots, q_n)$ and retain the same form except for the third one, which becomes

$$\lim_{\lambda\to\infty}\int_{V_\lambda}[F(x_1 - a - y) - F(-y)]c_\rho(y)\,dy$$

$$= \lim_{\lambda\to\infty}\int_{V_\lambda}[F(x_1 - a - y) - F(-y - a)]c_\rho(y)\,dy$$

$$+ \lim_{\lambda \to \infty} \int_{V_\lambda} [F(-a-y) - F(-y)] c_\rho(y)\, dy \tag{29}$$

$$= \lim_{\lambda \to \infty} \int_{V_\lambda + a} [F(x_1 - y) - F(-y)] c_\rho(y-a)\, dy$$

$$+ \lim_{\lambda \to \infty} \int_{V_\lambda} [F(-a-y) - F(-y)] c_\rho(y)\, dy \tag{30}$$

All limits in (29) and (30) exist by assumption. We see from (30) that the translated state $\tau_a \rho_{E,(V_\lambda)}$ obeys the same BBGKY equation (19), but with E, (V_λ) replaced by E_a given by (27) and $(V_\lambda + a)$, thus proving (i). Parts (ii)–(iv) are proved in a similar way. ∎

Corollary. Let $\rho_{E,(V_\lambda)}$ be a regular equilibrium state. If $|x|^{\nu-1}|\nabla F_\alpha(x)| = o(1)$, then $\tau_a \rho_{E,(V_\lambda)}$ is a RE state with respect to E_a, (V_λ).

Proof. If $|x|^{\nu-1}|\nabla F_\alpha(x)| = o(1)$, then $\rho_{E,(V_\lambda)} = \rho_{E,(V_\lambda + a)}$ by Proposition 2(iv) and thus $\tau_a \rho_{E,(V_\lambda)} = \rho_{E_a,(V_\lambda + a)} = \rho_{E_a,(V_\lambda)}$. ∎

Proposition 4. Let $\rho_{E,(V_\lambda)}$ be a RE state.

(i) $\rho_{E,(V_\lambda)}$ invariant under a translation a implies

$$\lim_{\lambda \to \infty} \int_{V_\lambda} dy\, [F(-a-y) - F(-y)] c_\rho(y) = 0 \tag{31}$$

(ii) $\rho_{E,(V_\lambda)}$ invariant under space inversion implies $E = 0$.
(iii) $\rho_{E,(V_\lambda)}$ invariant under a rotation R such that $RE \neq E$ implies $E = 0$.
(iv) $\rho_B = 0$ and $\rho_{E,(V_\lambda)}$ invariant under charge conjugation implies $E = 0$.

Proof. Part (i). Setting $\rho^{(n)}_{\sigma_1 \ldots \sigma_n}(x_1 - a, \ldots, x_n - a) = \rho^{(n)}_{\sigma_1 \ldots \sigma_n}(x_1, \ldots, x_n)$ in the BBGKY equation (19), we get

$$\lim_{\lambda \to \infty} \int_{V_\lambda} [F(x-a-y) - F(-y)] c_\rho(y)\, dy$$

$$= \lim_{\lambda \to \infty} \int_{V_\lambda} [F(x-y) - F(-y)] c_\rho(y)\, dy$$

i.e.,

$$\lim_{\lambda \to \infty} \int_{V_\lambda} [F(x-a-y) - F(x-y)] c_\rho(y)\, dy = 0$$

we get (31) by setting $x = 0$.

Parts (ii) and (iii). Let $\mathcal{O}$ be a rotation or an inversion; $\rho^{(1)}(x, \sigma) = \rho^{(1)}(\mathcal{O}x, \sigma)$ implies $\nabla\rho^{(1)}(x, \sigma) = \mathcal{O}^{-1}\nabla\rho^{(1)}(\mathcal{O}x, \sigma)$, and thus

$$E + \lim_{\lambda\to\infty}\int_{V_\lambda}[F(x-y) - F(-y)]c_\rho(y)\,dy$$
$$= \mathcal{O}^{-1}\left\{E + \lim_{\lambda\to\infty}\int_{V_\lambda}[F(\mathcal{O}x-y) - F(-y)]c_\rho(y)\,dy\right\}$$

Setting $x = 0$, we obtain $E = \mathcal{O}E$.

Part (iv) follows from Proposition 3(iv).

Proposition 5. "Scaling property." Let $\rho_{(\beta,E,\rho\beta,(V_\lambda))}$ be a RE state. If the force satisfies the scaling transformation

$$F(lx) = l^{-\nu}F(x)$$

then the transformed state $\tau_l\rho$ defined by

$$(\tau_l\rho^{(n)})(q_1,\dots,q_n) = l^{n\nu}\rho^{(n)}_{\sigma_1\dots\sigma_n}(x_1\cdots x_n)$$

is a RE state with respect to the parameters

$$\beta^{(l)} = l^{1-\nu}\beta, \qquad E^{(l)} = l^\nu E, \qquad \rho_B^{(l)} = l^\nu\rho_B, \qquad V_\lambda^{(l)} = l^{-1}V_\lambda$$

Proof.

$$\nabla_1(\tau_l\rho^{(n)})(q_1,\dots,q_n)$$
$$= l^{n\nu}\,\nabla_1\rho^{(n)}_{\sigma_1,\dots,\sigma_n}(lx_1,\dots,lx_n)\cdot l$$
$$= \beta l^{1+n\nu}\sigma_1\left\{E + \lim_{\lambda\to\infty}\int_{V_\lambda}[F(lx_1-y) - F(-y)]c_\rho(y)\,dy\right.$$
$$\left.+\sum_{j=2}^{n}\sigma_jF(l(x_1-x_j))\right\}\rho^{(n)}_{\sigma_1\dots\sigma_n}(lx_1,\dots,lx_n)$$
$$+\beta l^{1+n\nu}\sigma_1\int_{\mathbb{R}^\nu}\left\{\sum_{\sigma\in\Sigma}\sigma F(lx_1-y)[\rho^{(n+1)}_{\sigma_1,\dots,\sigma_n,\sigma}(lx_1,\dots,lx_n,y)\right.$$
$$\left.-\rho^{(n)}_{\sigma_1,\dots,\sigma_n}(lx_1,\dots,lx_n)\rho^{(1)}_\sigma(y)]\right\}dy$$
$$= \beta l\sigma_1\left(E + l^{-\nu}\lim_{\lambda\to\infty}\int_{l^{-1}V_\lambda}\left\{[F(x_1-y) - F(-y)]\left[\sum_{\sigma\in\Sigma}\sigma(\tau_l\rho^{(1)})(y\sigma)\right.\right.\right.$$
$$\left.\left.\left.-l^\nu\rho_B\right]\right\}dy + l^{-\nu}\sum_{j=2}^{n}\sigma_jF(x_1-x_j)\right)(\tau_l\rho^{(n)})(q_1,\dots,q_n)$$
$$+\beta l^{1-\nu}\int dq\,\{F(q_1,q)[\tau_l\rho^{(n+1)})(q_1,\dots,q_n,q)$$

$$
\begin{aligned}
&- (\tau_l\rho^{(n)})(q_1,\dots,q_n)(\tau_l\rho^{(1)})(q)]\} \\
&= \beta l^{1-\nu}\sigma_1\left\{\left[l^\nu E + \lim_{\lambda\to\infty}\int_{l^{-1}V_\lambda}\left\{dy\,[F(x_1 - y) - F(-y)]\right.\right.\right. \\
&\quad \left.\left.\times\left[\sum_{\sigma\in\Sigma}\sigma(\tau_l\rho^{(1)})(y\sigma) - l^\nu\rho_B\right]\right\} + \sum_{j=2}^{n}\sigma_j F(x_1 - x_j)\right](\tau_l\rho^{(n)})(q_1,\dots,q_n)\Big\} \\
&\quad + \beta l^{1-\nu}\int dq\,\{F(q_1, q)[(\tau_l\rho^{(n+1)})(q_1,\dots,q_n,q) \\
&\quad - (\tau_l\rho^{(n)})(q_1,\dots,q_n)(\tau_l\rho^{(1)})(q)]\}
\end{aligned}
$$

which concludes the proof. ∎

Corollary. If the force has the scaling property and if $\rho_B \neq 0$ the RE states are essentially parametrized by E, (V_λ), and the *plasma parameter* $\gamma^{(\nu)}$, where

$$\gamma^{(\nu)} = e^2\beta(\rho_\beta)^{(\nu-1)/\nu}$$

Indeed by a scaling transformation we can always take $\rho_B^{(l)} = 1$ and the RE is thus only parametrized by the temperature $\beta^{(l)} = \gamma^{(\nu)}(1/e^2)$, E, and (V_λ).

The corollary extends to the state the scaling property that has already been established for the thermodynamic quantities.[12]

3.3. Interpretation of *E*

We conclude this section by a discussion of the interpretation of the parameter E and of the effective field $E_\rho(x)$ of (18) assuming that the force satisfies (iv) of Proposition 2.

We emphasize again that a RE state $\rho_{E,(V_\lambda)}$ is also a RE state with respect to the *same parameter* E and any *translated sequence of regions* $(V_\lambda + a)$. Furthermore, the translated state $\tau_a\rho_{E,(V_\lambda)}$ is a RE state with respect to the *same sequence* (V_λ) and the *new parameter* E_a given by (27).

Finally, (28) shows that $E_\rho(x)$ transforms in a covariant manner under the translations, and $E = E_\rho(0)$, $E_a = E_{\tau_a\rho}(0) = E_\rho(-a)$. In particular, if the state is periodic, E_a is a periodic function of a. Thus the role of the parameters E and (V_λ) in the search for periodic states can be described as follows: the asymptotic shape of the region V_λ will determine the shape of the fundamental cell, whereas E will locate the fundamental cell in space (up to the points a in the fundamental cell where the effective field E_a has the same value).

Moreover, we should add that for Coulomb systems $E_\rho(x)$ is the solution of the electrostatic equation div $E_\rho(x) = \omega_\nu c_\rho(x)$ with the boundary condition $E_\rho(0) = E$ [or div $E_\rho(x) = \omega_\nu \int g(x - y)c_\rho(y)\,dy$, if we have the smeared Coulomb force (6)]. *Thus $E_\rho(x)$ is the electric field at point x in the infinite* *system.* These features can be illustrated on one-dimensional Coulomb systems. For these systems, one can construct explicitly RE states by means of thermodynamic limits of finite Gibbs states; these states have the following properties:

(1) The one-dimensional jellium has a RE state which is nontrivially periodic with period ρ_B^{-1} and invariant under inversion around the origin.[5,7] For this state, $E = 0$ by Proposition 4(ii), and

$$E_\rho(x) = \lim_{\substack{L_1 \to -\infty \\ L_2 \to \infty}} \int_{L_1}^{L_2} [\operatorname{sign}(x - y) - \operatorname{sign}(-y)] c_\rho(y)\, dy = 2 \int_0^x c_\rho(y)\, dy \tag{32}$$

represents the electric field at the point x. (Here the limit is independent of the sequence of intervals $[L_1, L_2]$ because in one space dimension, shape dependence is not possible!).

(2) The one-dimensional, two-component Coulomb gas has a RE state which is invariant under translations and charge conjugation.[5,6] This state is necessarily neutral and by Proposition 4(iii) it has zero effective field,

$$E_\rho(x) = E_{\tau_{-x}\rho}(0) = E_\rho(0) = E = 0$$

It can be shown[9] that this system also has translation-invariant neutral RE states with nonzero electric field, i.e., with $E_\rho(x) = E \neq 0$.

By Proposition 4(ii) and 4(iv) it is clear that these states cannot be invariant under space inversion or charge conjugation.

We shall see in the last section that the structure of the equilibrium equations imposes a bound on the possible values of E, the bound being independent of the state. This confirms that E is not an external applied electric field (which could be chosen arbitrarily), but the effective field in the system.

4. LOCAL NEUTRALITY OF REGULAR EQUILIBRIUM STATES AND CONSEQUENCES

It follows from Proposition 4 that any RE state that is $\mathbb{R}^\nu$-invariant, i.e., $\tau_a \rho_E = \rho_E$ for all a in $\mathbb{R}^\nu$, must satisfy

$$c_\rho \left\{ \operatorname*{Lim}_{\lambda \to \infty} \int_{V_\lambda} dy\, [F(-a - y) - F(-y)] \right\} = 0, \qquad \forall a \in \mathbb{R}^\nu \tag{33}$$

with $c_\rho = \sum \sigma\rho_\sigma - \rho_B$, $\rho_\sigma = \rho^{(1)}(x\sigma)$. This yields immediately $c_\rho = 0$, i.e., the neutrality of the state, as soon as the force satisfies the condition

$$\operatorname*{Lim}_{\lambda \to \infty} \int_{V_\lambda} dy\, [F(-a - y) - F(-y)] \neq 0 \qquad \text{or does not exist} \tag{34}$$

We notice at once that a necessary condition for us to be able to draw a conclusion about neutrality is that the force decreases slowly enough at infinity; in particular, if the force is $\mathscr{L}^1$, the limit (34) is zero and we cannot draw a conclusion. In fact, we shall see that the condition (34) is satisfied as soon as the force decreases at infinity like the Coulomb force or slower.

The next proposition shows that this result remains valid for states that are not necessarily $\mathbb{R}^\nu$-invariant.

Proposition 6. Assume that the force satisfies the following conditions

$$F(x) = F'(x) + F''(x) \tag{35}$$

with $F'(x)$ in $\mathscr{L}^1$ and $F''(x) = -\nabla\phi(x)$, ϕ of class C^2, and

$$\operatorname*{Lim}_{\lambda\to\infty} \lambda^\gamma F''(\lambda\hat{x}) = d(\hat{x}), \qquad \hat{x} = x/|x|, \qquad \gamma \geqslant 0 \tag{36}$$

where $d(\hat{x})$ is not identically zero,

$$|x|^{\gamma+1}|\nabla F''_\alpha(x)| = O(1) \qquad \text{as} \quad |x| \to \infty \tag{37}$$

(i) Any $\mathbb{R}^\nu$-invariant, RE state ρ_{E,V_0} is *locally neutral*, i.e.,

$$\bar{c}_\rho = \sum_\sigma \sigma\bar{\rho}_\sigma - \rho_B = \lim_{\Lambda\to\mathbb{R}^\nu} \frac{1}{|\Lambda|}\int_\Lambda c_\rho(y)\,dy = 0$$

if the force is such that $\gamma \leqslant \nu - 1$ and satisfies

$$\int_{\partial V_0} \frac{d(-\hat{x})\cdot a}{|x|^\gamma}\,ds \neq 0 \tag{38}$$

for some a in $\mathbb{R}^\nu$, where ∂V_0 is the boundary of V_0.

(ii) Let $\mathscr{T}$ be a discrete subgroup of the translation group $\mathbb{R}^\nu$ generated by $\{e_1,\ldots, e_\nu\}$ and Δ_0 be the unit cell based on $\{e_\alpha\}$. Any regular equilibrium state ρ_{E,Δ_0} invariant under $\mathscr{T}$ is *locally neutral*, i.e.,

$$\bar{c}_\rho = \frac{1}{\Delta_0}\int_{\Delta_0} c_\rho(y)\,dy = \lim_{\Lambda\to\mathbb{R}^\nu}\frac{1}{|\Lambda|}\int_\Lambda c_\rho(y)\,dy = 0$$

if the force is such that $\nu - 2 < \gamma \leqslant \nu - 1$ and satisfies the condition (38) for some a in $\mathscr{T}$.

Corollary 1. (i) The $\mathbb{R}^\nu$-invariant RE states ρ_{E,V_0} of Coulomb systems are locally neutral.

(ii) Assume that $\gamma \leqslant \nu - 1$ and that the force is asymptotically radial $[d(\hat{x}) = \text{cst}\,\hat{x}]$. Any $\mathbb{R}^\nu$-invariant RE state $\rho_{E;V_0}$ with V_0 convex is neutral.

Proof of Corollary 1. This is a straightforward consequence of the proposition. To have (38) it is sufficient to show in both cases that

$$\int_{\partial V_0} \frac{\hat{x}_\alpha}{|x|^\gamma}\,ds_\alpha \neq 0 \qquad \text{for some} \quad \alpha = 1,\ldots,\nu \tag{39}$$

But in the Coulomb case, $\gamma = \nu - 1$ and we have that

$$\int_{\partial V_0} \frac{\hat{x}}{|x|^{\nu-1}} \cdot ds = \omega_\nu \neq 0$$

for any V_0, where ω_ν is the surface of a sphere of radius 1 in $\mathbb{R}^\nu$.

In the second case V_0 convex implies $\hat{x} \cdot ds \geqslant 0$ on ∂V_0 and therefore again $\int_{\partial V_0} (\hat{x}/|x|^\gamma) \cdot ds > 0$. ■

Corollary 2. Assume that the force is asymptotically radial, and let ρ_{E,V_0} be an $\mathbb{R}^\nu$-invariant RE state with V_0 convex. Then ρ_{E,V_0} is also a RE state with respect to any sequence (V_λ').

Proof of Corollary 2. If $\gamma > \nu - 1$, the result follows from Proposition 1(ii).

If $\gamma \leqslant \nu - 1$, the state is neutral by Corollary 1 above. Therefore

$$\int_{V_\lambda'} [F(x - y) - F(-y)]c_\rho(y)\, dy = 0$$

for any sequence (V_λ'). The term containing the (V_λ') dependence in (19) drops out and thus ρ_{E,V_0} is a solution of (19) for any sequence (V_λ'). ■

Corollary 2 shows that irrespective of the nature of the force, an $\mathbb{R}^\nu$-invariant RE state is always shape independent, as is intuitively expected.

Corollary 3. Consider a system with $\rho_B = 0$ and all charges of the same sign. Under the conditions of Proposition 6, the system has no RE states with nonvanishing density.

Indeed $\bar{c}_\rho = \sum_{\sigma \in \Sigma} \sigma \bar{\rho}_\sigma = 0$ implies $\bar{\rho}_\sigma = 0, \forall \sigma \in \Sigma$. ■

In particular, the corollary shows that there does not exist $\mathbb{R}^\nu$ or $\mathscr{T}$-invariant equilibrium states for one-component systems with $\rho_B = 0$ if the force decreases like the Coulomb force or slower.

The proof of Proposition 6 is based on two results which we shall now establish.

Lemma 2. Let $\mathscr{T}$ be a discrete subgroup of the translation group $\mathbb{R}^\nu$ generated by $\{e_1, \ldots, e_\nu\}$, and (V_λ) be a sequence of volumes defined as the union of unit cells

$$\Delta_n = \left\{ x + \sum_{i=1}^{\nu} n_i e_i; \quad x \in \Delta_0, \quad n \in \mathbb{Z}^\nu \right\}$$

If the force satisfies the conditions

$$|x|^{\nu-1} |\nabla F_\alpha(x)| = o(1) \qquad \text{as} \quad |x| \to \infty \tag{40}$$

and

$$\lim_{\lambda\to\infty}\int_{V_\lambda}[F_\alpha(-a-y)-F_\alpha(-y)]\,dy \neq 0 \qquad \begin{array}{l}\text{or does not exist}\\ \text{for some } a \text{ in } \mathscr{T}\end{array} \tag{41}$$

then any regular equilibrium state $\rho_{E,(V_\lambda)}$ that is invariant under $\mathscr{T}$ is *locally neutral*, i.e.,

$$\bar{c}_\rho = \lim_{\lambda\to\infty}\frac{1}{|\Lambda|}\int_\Lambda c_\rho(y)\,dy = \frac{1}{|\Delta_0|}\int_{\Delta_0} c_\rho(y)\,dy = 0$$

Proof. Since $\tau_a\rho_{E,(V_\lambda)} = \rho_{E,(V_\lambda)}$ for a in $\mathscr{T}$, it follows from Proposition 4 that

$$\lim_{\lambda\to\infty}\int_{V_\lambda}[F_\alpha(-a-y)-F_\alpha(-y)]c_\rho(y)\,dy = 0 \tag{42}$$

Let us denote by $\bar{c}_\rho$ and $\bar{F}_{\alpha,n}$ the average charge density and the average force in Δ_n:

$$\bar{c}_\rho = \frac{1}{|\Delta_0|}\int_{\Delta_0} c_\rho(y)\,dy, \qquad \bar{F}_{\alpha,n} = \frac{1}{|\Delta_n|}\int_{\Delta_n} F_\alpha(-y)\,dy$$

Since $c_\rho(y+a) = c_\rho(y)$, $a\in\mathscr{T}$, we have

$$\begin{aligned}\int_{V_\lambda}&[F_\alpha(-a-y)-F_\alpha(-y)]c_\rho(y)\\ &= \bar{c}_\rho\int_{V_\lambda}[F_\alpha(-a-y)-F_\alpha(-y)]\,dy\\ &\quad + \sum_{\Delta_n\subset(V_\lambda+a)}\int_{\Delta_n}[F_\alpha(-y)-\bar{F}_{\alpha,n}]c_\rho(y)\,dy\\ &\quad - \sum_{\Delta_n\subset V_\lambda}\int_{\Delta_n}[F_\alpha(-y)-\bar{F}_{\alpha,n}]c_\rho(y)\,dy\end{aligned} \tag{43}$$

where $V_\lambda + a = \{x + a; x\in V_\lambda\}$.

We shall show that condition (40) implies that the difference of the last two terms of (43) tends to zero as $\lambda\to\infty$; from (42) it thus follows that

$$\bar{c}_\rho\left\{\lim_{\lambda\to\infty}\int_{V_\lambda}[F_\alpha(-a-y)-F_\alpha(-y)]\,dy\right\} = 0$$

which concludes the proof of this lemma.

To show that the difference of the last two terms of (43) tends to zero as $\lambda\to\infty$, we note that this difference is majorized by

$$\|c_\rho\|_\infty \sum_{\Delta_n\subset V_\lambda\Delta(V_\lambda+a)}\int_{\Delta_n}|F_\alpha(-y)-\bar{F}_{\alpha,n}|\,dy \tag{44}$$

But if $x, y \in \Delta_n \subset V_\lambda \triangle (V_\lambda + a)$ we have for λ large enough as in (24)

$$|F_\alpha(y) - F_\alpha(x)| \leqslant \text{cst}\, A/(R_\lambda^-)^{\nu-1}$$

where R_λ^- is the radius of the ball $B_\lambda^- \subset V_\lambda$, and also

$$|F_\alpha(-y) - \bar{F}_{\alpha,n}| \leqslant \frac{1}{|\Delta_n|}\int_{\Delta_n} |F_\alpha(-y) - F_\alpha(-x)|\, dx \leqslant \text{cst}\, \frac{A}{(R_\lambda^-)^{\nu-1}}$$

Thus

$$\sum_{\Delta_n \subset V_\lambda \triangle (V_\lambda + a)} \int_{\Delta_n} |F_\alpha(-y) - \bar{F}_{\alpha,n}|\, dy \leqslant \text{cst}\, A\, \frac{|(V_\lambda + a) \triangle V_\lambda|}{(R_\lambda^-)^{\nu-1}} \leqslant \text{cst}\, A \tag{45}$$

where we have used the properties (b) and (c) of the sequence (V_λ) as in (26) of part (iv) of Proposition 2. The result follows from the fact that A can be chosen arbitrarily small. ∎

It just remains to study the forces for which condition (41) will be satisfied, i.e., for what forces it is possible to prove local neutrality.

Lemma 3. Let $(V_\lambda) = (\{\lambda x; x \in V_0\})$ be a sequence of volumes obtained by dilatation of V_0.

(i) If the force satisfies conditions (35)–(37) of Proposition 6, then

$$\lim_{\lambda\to\infty} \lambda^{\gamma-\nu+1} \int_{V_\lambda} [F''_\alpha(y + a) - F''_\alpha(y)]\, dy = \int_{\partial V_0} \frac{d(\hat{x})\cdot a}{|x|^\gamma}\, ds_\alpha$$

(ii) If in addition (38) is satisfied, then

$$\lim_{\lambda\to\infty} \left| \int_{V_\lambda} [F(-a - y) - F(-y)]\, dy \right| = \begin{cases} 0 & \text{if } \gamma > \nu - 1 \\ \left| \int_{\partial V_0} \dfrac{d(-\hat{x})\cdot a}{|x|^{\nu-1}}\, ds \right| & \text{if } \gamma = \nu - 1 \\ \infty & \text{if } \gamma < \nu - 1 \end{cases}$$

Proof. (i) Using successively Gauss' theorem, a scaling transformation, and the Taylor theorem, we have

$$\begin{aligned} &\lambda^{\gamma-\nu+1} \int_{V_\lambda} [F''_\alpha(y + a) - F''_\alpha(y)]\, dy \\ &= -\lambda^{\gamma-\nu+1} \int_{V_\lambda} [(\partial_\alpha\phi)(y + a) - (\partial_\alpha\phi)(y)]\, dy \\ &= -\lambda^{\gamma-\nu+1} \int_{\partial V_\lambda} [\phi(y + a) - \phi(y)]\, ds_\alpha \\ &= -\lambda^\gamma \int_{\partial V_0} [\phi(\lambda x + a) - \phi(\lambda x)]\, ds_\alpha \\ &= \lambda^\gamma \int_{\partial V_0} [a\cdot F''(\lambda x + \theta a)]\, ds_\alpha, \qquad 0 \leqslant \theta \leqslant 1 \end{aligned} \tag{46}$$

Since $x \in \partial V_0$ is nonzero, we can take λ large enough to replace the force in (46) by its asymptotic behavior. According to (36) and (37), we have

$$\lambda^\gamma F_\alpha(\lambda x + \theta a) = \lambda^\gamma \frac{F_\alpha(\lambda x)}{|x|^\gamma} + o(1)$$

and thus the result follows by dominated convergence.

(ii) The proof is immediate using (i) and the fact that $F'(x)$ is $\mathscr{L}^1$. ■

Lemmas 2 and 3 give the proof of Proposition 6.

Remark 1. The equilibrium equation (19) imposes the neutrality of the state only if the force decreases as the Coulomb force or slower. If the force decreases faster than Coulomb, nothing can be concluded about neutrality.

Remark 2. It should be stressed that condition (40) is only necessary for $\mathscr{T}$-invariant states which are not $\mathbb{R}^\nu$-invariant. For such states condition (40) imposes that the force should decrease faster than $1/|x|^{\nu-2}$.

Remark 3. Part (i) of Proposition 6 remains valid under the weaker condition $|x|^\gamma|\nabla F_\alpha(x)| = o(1)$ instead of (37).

To conclude the discussion of neutrality, we shall now prove a stronger result for one-dimensional systems: namely that *any* regular equilibrium state is locally neutral if the force does not vanish at infinity.

Proposition 7. Consider a one-dimensional system with the two-body force

$$F(x) = d \operatorname{sign} x + \varphi(x) \qquad \text{with} \quad \varphi \in \mathscr{L}^1, \quad d \neq 0$$

Then any regular equilibrium state satisfying the conditions:

(i) $\int dy\, |\rho_T^{(2)}(x, y)|$ is uniformly bounded

(ii) $\operatorname{Lim}_{L\to\infty}(1/L) \int_0^L dy\, c_\rho(y) = \bar{c}_\rho$ exists

is either locally neutral, i.e., $\bar{c}_\rho = 0$, or the particles have zero charge density, i.e., $\bar{c}_\rho + \rho_B = 0$.

Proof. In this case Eq. (19) yields

$$\frac{\partial}{\partial x}\left(\sum_\sigma \rho^{(1)}(x\sigma)\right) = \beta E_\rho(x)[c_\rho(x) + \rho_B] + \beta \int dy\, F(x - y) \sum_{\sigma\bar\sigma} \sigma\bar\sigma \rho_T^{(2)}(x\sigma, y\bar\sigma)$$

with

$$E_\rho(x) = 2d \int_0^x c_\rho(y)\, dy + \int [\varphi(x - y) - \varphi(-y)] c_\rho(y)\, dy$$

Apply $(1/L^2) \int_0^L dx$ to both sides of this equation and take the limit $L \to \infty$; the second term on the right-hand side tends to zero because of (i) and we obtain

$$0 = \lim_{L\to\infty} \frac{1}{L^2} \int_0^L E_\rho(x)[c_\rho(x) + \rho_B]\, dx$$

$$= \lim_{L\to\infty} \frac{1}{L^2} \int_0^L \left\{ [c_\rho(x) + \rho_B] \int_0^x 2dc_\rho(y)\, dy \right\} dx$$

$$= \lim_{L\to\infty} d\left\{ \left[\frac{1}{L}\int_0^L c_\rho(x)\, dx\right]^2 + \frac{2\rho_B}{L^2}\int_0^L dx \int_0^x dy\, c_\rho(y) \right\}$$

which gives, using l'Hospital's theorem,

$$0 = \bar{c}_\rho{}^2 + \rho_B \lim_{L\to\infty} \frac{1}{L}\int_0^L c_\rho(y)\, dy = \bar{c}_\rho(\bar{c}_\rho + \rho_B) \quad \blacksquare$$

Remark. For the one-dimensional jellium this theorem describes two situations: either the density of charge $\bar{c}_\rho$ is zero or the density of particles vanishes.

5. THE CANONICAL SUM RULES

In this section, we show that if the force has infinite range and if the state has a rate of clustering which is faster than the decrease of the force, then the correlation functions obey a new hierarchy of constraints. These constraints link the n-point to the $(n + 1)$-point correlation functions and we call them the *canonical sum rules*. The interpretation and some consequences of the canonical sum rules will be discussed in the last section. The simplest sum rule is the following relation between the one- and the two-point correlation functions:

$$0 = \sigma_1 \rho^{(1)}(x_1\sigma_1) + \sum_\sigma \int \sigma[\rho^{(2)}(x_1\sigma_1, x\sigma) - \rho^{(1)}(x_1\sigma_1)\rho^{(1)}(x\sigma)]\, dx \tag{47}$$

or in a more condensed notation

$$\sigma_1 \rho^{(1)}(q_1) + \int dq\, \sigma \rho_T^{(2)}(q_1, q) = 0 \tag{48}$$

where $\rho_T^{(2)}(q_1, q_2)$ is the two-point truncated correlation function.

The general form of the sum rule is

$$0 = \left(\sum_{j=1}^n \sigma_j\right) \rho^{(n)}(q_1, \dots, q_n)$$

$$+ \int dq\, \sigma[\rho^{(n+1)}(q_1, \dots, q_n, q) - \rho^{(1)}(q)\rho^{(n)}(q_1, \dots, q_n)] \tag{49}$$

with $n \geqslant 1$. This relation can also be written in terms of the truncated correlation functions

$$\left(\sum_{j=1}^{n} \sigma_j\right) \rho_T^{(n)}(q_1,\dots, q_n) + \int dq\, \sigma \rho_T^{(n+1)}(q_1,\dots, q_n, q) = 0 \tag{50}$$

The equivalence between (49) and (50) is shown in the Appendix.

We establish now that, under some additional assumptions on the clustering, a RE state has to obey the canonical sum rules. The precise statement is formulated in Proposition 8 and the proof is based on an asymptotic analysis of the BBGKY hierarchy (19) as $|x_1| \to \infty$. We shall then give an extension of Proposition 8 which is valid for infinite-range but integrable potentials.

Proposition 8. Assume that the two-body force $F(x)$ satisfies the conditions:

(a1) $\lim_{\lambda\to\infty} \lambda^\gamma F(\lambda \hat{x}) = d(\hat{x})$, $\hat{x} = x/|x|$, $\gamma \geqslant 0$, where $d(\hat{x})$ is a bounded function of $\hat{x}$ which is not identically zero.

(a2) $|x|^\gamma |\nabla F(x)| = o(1)$, $|x| \to \infty$.

Then any RE state $\rho_{E,(V_\lambda)}$ satisfying conditions (b) and (c) stated below obeys the sum rules (49) and (50).

(b1) $E_\rho(x)$ is uniformly bounded in $\mathbb{R}^\nu$.

(b2) There exist $r > 0$ and $\rho_0 > 0$ such that $\int_{B(x,r)} \rho^{(1)}(y\sigma)\, dy > \rho_0$ for $|x|$ large enough, where $B(x, r)$ is the ball of center x and radius r.

The correlation functions $\rho^{(n)}(q_1,\dots, q_n)$ of $\rho_{E,(V_\lambda)}$ have the following clustering properties.

(c1) For x_2 fixed

$$\rho_T^{(2)}(x_1\sigma_1, x_2\sigma_2) = \begin{cases} o(1/|x_1|^\gamma) & \text{if } \gamma > \nu \\ O(1/|x_1|^{\nu+\epsilon}) & \text{if } \gamma \leqslant \nu, \quad \epsilon > 0 \end{cases}$$

(c2) For $n \geqslant 3$ and $x_3,\dots, x_n$ fixed

$$\rho_T^{(n)}(x_1\sigma_1, x_2\sigma_2, x_3\sigma_3,\dots, x_n\sigma_n) = \begin{cases} o(1/|x_1|^\gamma) & \text{if } \gamma > \nu \\ O(1/|x_1|^{\nu+\epsilon}) & \text{if } \gamma \leqslant \nu \end{cases}$$

uniformly in x_2, and

$$\text{(c3)} \quad \lim_{|x_2|\to\infty} \int |\rho_T^{(n)}(x_1\sigma_1, x_2\sigma_2, x_3\sigma_3,\dots, x_n\sigma_n)|\, dx_1 = 0, \qquad n \geqslant 3$$

Remarks. Condition (b1), i.e., the effective field is uniformly bounded, is obviously verified if the state is periodic: then $E_\rho(x)$ is a bounded, periodic function of x; (b2) means that the local density of particles does not vanish as $|x| \to \infty$. Again (b2) is true if the state is periodic (choose r large enough compared to a linear dimension of the fundamental cell). Conditions (c1) and (c2) ensure that the clustering is faster than the decrease of the force when

$\gamma > \nu$, and the integrability of the truncated correlation functions when $\gamma \leqslant \nu$.

To establish the proposition, we shall take the following steps:

(i) We express the correlation functions occurring in the integral of Eq. (19) in terms of the truncated ones (see Lemma 4 below).

(ii) We integrate the variable x_1 in both parts of the BBGKY equation (19) on the ball $B(\lambda\hat{u}, r)$, where $\hat{u}$ is chosen such that $d(\hat{u}) \neq 0$, λ is large enough, and r is given in assumption (b2). [With this, we eliminate the derivative on the right-hand side of Eq. (19); no assumption has been made here on the decrease of the derivatives of the truncated correlation functions.]

(iii) We multiply the whole equation by λ^γ and take the limit $\lambda \to \infty$.

To abbreviate the notation we write simply for $n \geqslant 2$

$$\rho^{(n)}(q_1 q_2 \cdots q_n) = \rho(q_1 Q) \tag{51}$$

$$\rho^{(n+1)}(q q_1 q_2 \cdots q_n) = \rho(q q_1 Q) \tag{52}$$

where Q is the set of arguments $(q_2 \cdots q_n)$.

Lemma 4.

$$\rho(qq_1Q) - \rho(q)\rho(q_1Q)$$
$$= \rho_T(qq_1)\rho(Q) + \rho(q_1)[\rho(qQ) - \rho(q)\rho(Q)] + R(qq_1Q) \tag{53}$$

with

$$R(qq_1Q) = \sum_{\substack{Q_1, Q_2 \subset Q \\ Q_1 \cap Q_2 = \varnothing, Q_1 \neq \varnothing, Q_2 \neq \varnothing}} \rho_T(qQ_1)\rho_T(q_1Q_2)\rho(Q \backslash Q_1 Q_2)$$
$$+ \sum_{\varnothing \neq Q_1 \subset Q} \rho_T(qq_1Q_1)\rho(Q\backslash Q_1) \tag{54}$$

The sums in (54) run on the subsets Q_1, Q_2 of Q as indicated and $Q\backslash Q_1Q_2$ is the difference of the sets Q and $Q_1 \cup Q_2$ $[\rho(\varnothing) = 1]$. The proof is given in the Appendix.

The explicit form of $R(qq_1Q)$ will be irrelevant in the following. The only point that matters is that $R(qq_1Q)$ is a finite sum of truncated functions, where the arguments q or q_1 [or the pair (qq_1)] occur always in conjunction with some other argument $q_j \in Q$, $j = 2,\ldots n$, so that the clustering properties (c1), (c2), and (c3) apply to the terms of $R(qq_1Q)$.

We insert the expression (53) given in Lemma 4 into the last integral on the right-hand side of Eq. (19) (with $n > 2$).

Using the notation introduced in (52), we get

$$\beta^{-1} \nabla_1[\rho(q_1Q) - \rho(q_1)\rho(Q)] \tag{55}$$

$$= \sigma_1 E_\rho(x_1)[\rho(q_1Q) - \rho(q_1)\rho(Q)] \tag{55a}$$

$$+ \sum_{j=2}^{n} F(q_1 q_j)\rho(q_1 Q) \tag{55b}$$

$$+ \rho(q_1)\int dq\, F(q_1 q)[\rho(qQ) - \rho(q)\rho(Q)] \tag{55c}$$

$$+ \int dq\, F(q_1 q)R(qq_1 Q) \tag{55d}$$

In getting (55), we have also used Eq. (19) in the case $n = 2$ to express the integral on the two-point truncated correlation function as

$$\beta \int F(q_1 q)\rho_T(q_1 q)\, dq = \nabla_1 \rho(q_1) - \beta\sigma_1 E_\rho(x_1)\rho(q_1)$$

We study now the behavior of the various terms of Eq. (55) as $|x_1| \to \infty$ and show that (55b) and (55c) will give rise to the sum rules.

More precisely, we set

$$x_1 = \lambda\hat{u} + y, \qquad |\hat{u}| = 1 \tag{56}$$

with $\hat{u}$ such that $d(\hat{u}) \neq 0$ and $|y| \leqslant r$. Then we multiply Eq. (55) by λ^γ and determine the asymptotic behavior of the terms (55a)–(55d) as $\lambda \to \infty$, keeping $\hat{u}$ and y fixed.

Finally, we perform the integration over the ball $B(\lambda\hat{u}, r)$ as indicated in (ii) and take the limit $\lambda \to \infty$.

To simplify the discussion of Eq. (55), we define the function

$$g_\sigma(x) = \rho(x\sigma, Q) - \rho(x\sigma)\rho(Q) \tag{57}$$

the arguments in Q being held fixed ($Q \neq \varnothing$).

Notice that the clustering assumptions (c1) and (c2) imply

$$g_\sigma(x) = \begin{cases} o(1/|x|^\gamma), & \gamma > \nu \\ O(1/|x|^{\nu+\epsilon}), & \gamma \leqslant \nu \end{cases} \qquad |x| \to \infty \tag{58a}$$

In particular, if $x_1 = \lambda\hat{u} + y$, $|\hat{u}| = 1$,

$$g_{\sigma_1}(x_1) = \begin{cases} o(1/\lambda^\gamma), & \gamma > \nu \\ O(1/\lambda^{\nu+\epsilon}), & \gamma \leqslant \nu \end{cases} \qquad \lambda \to \infty \tag{58b}$$

uniformly with respect to y for $|y| \leqslant r$.

We proceed now to the analysis of the terms (55a)–(55d).

Term (55a). The term (55a) reads $\lambda^\gamma E_\rho(x_1) g_{\sigma_1}(x_1)$, $x_1 = \lambda\hat{u} + y$. Since, by assumption (b1), $E_\rho(x)$ remains bounded as $|x| \to \infty$, (58b) implies obviously that this term tends to zero uniformly with respect to y, $|y| \leqslant r$, as $\lambda \to \infty$.

Term (55b). Adding and subtracting $\sum_{j=2}^{n} F_\alpha(q_1 q_j)\rho(q_1)\rho(Q)$ in (55b), we write (55b) in the form

$$\sigma_1 \lambda^\gamma \sum_{j=2}^{n} \sigma_j F(x_1 - x_j)\rho(x_1\sigma_1)\rho(Q) + \sigma_1\lambda^\gamma \sum_{j=2}^{n} F(x_1 - x_j) g_{\sigma_1}(x_1) \quad (59)$$

with $x_1 = \lambda\hat{u} + y$.

From the asymptotic behavior of the force, we deduce that for fixed x,

$$F(x_1 - x) = F(\lambda\hat{u} + y - x) = F(\lambda\hat{u}) + o(1/\lambda^\gamma)$$

and thus

$$\lim_{\lambda\to\infty} \lambda^\gamma F(\lambda\hat{u} + y - x) = d(\hat{u}) \quad (60)$$

$$\lambda^\gamma F(\lambda\hat{u} + y - x) = O(1), \qquad \lambda \to \infty \quad (61)$$

uniformly with respect to y, $|y| \leqslant r$.

Since $\rho(\lambda x, \sigma)$ is uniformly bounded in λ, (60) implies that the first term of (59) converges to

$$\left(\sum_{j=2}^{n} \sigma_j\right)\rho(Q)[\sigma_1 d(\hat{u})\rho(\lambda\hat{u} + y, \sigma_1)] \quad (62)$$

as $\lambda \to \infty$.

By (58b) and (61), the second term of (59) is $o(1/\lambda^\gamma)$ and vanishes as $\lambda \to \infty$. Thus (62) is the asymptotic behavior of the term (55b) as $\lambda \to \infty$.

Term (55c). The term (55c) is

$$\sigma_1\rho(x_1\sigma_1)\lambda^\gamma \sum_\sigma \sigma \int F(x_1 - x) g_\sigma(x)\, dx, \qquad x_1 = \lambda\hat{u} + y \quad (63)$$

We show in the Appendix the following lemma:

Lemma 5. Under the assumptions (a1) and (a2) on $F(x)$ and (58) on the function $g_\sigma(x)$, one has

$$\lim_{\lambda\to\infty} \lambda^\gamma \int F(\lambda\hat{u} + y - x) g_\sigma(x)\, dx = d(\hat{u}) \int g_\sigma(x)\, dx \quad (64)$$

uniformly with respect to y, $|y| \leqslant r$.

Therefore (63) converges to

$$[\sigma_1 d(\hat{u})\rho(\lambda\hat{u} + y, \sigma_1)] \sum_\sigma \sigma \int g_\sigma(x)\, dx$$
$$= [\sigma_1 d(\hat{u})\rho(\lambda\hat{u} + y, \sigma_1)] \int dq\, \sigma[\rho(qQ) - \rho(q)\rho(Q)] \quad (65)$$

as $\lambda \to \infty$.

Term (55d). By a slight extension of Lemma 5, one shows (see Appendix) that

$$\lim_{\lambda\to\infty} \lambda^\gamma \left| \sigma_1 \sum_\sigma \sigma \int F(x_1 - x) R(x\sigma, x_1\sigma_1, Q)\, dx \right|$$
$$\leqslant \text{cst} \sum_\sigma \sigma \lim_{\lambda\to\infty} \int |R(x\sigma, x_1\sigma_1, Q)|\, dx, \qquad x_1 = \lambda\hat{u} + y \tag{66}$$

In view of the structure of $R(qq_1Q)$ (see Lemma 4) and the assumptions (c1)–(c3) on the clustering, this last limit is zero. Indeed for the terms in $R(qq_1Q)$ that involve products $\rho_T(qQ')\rho_T(q_1Q'')$ the limit is obviously zero. By assumption (c3), the limit is also zero for the terms in $R(qq_1Q)$ that involve $\rho_T(qq_1Q')$.

We now perform the integration of Eq. (55) over the ball $B(\lambda\hat{u}, r)$, i.e., over the variable y in the domain $|y| \leqslant r$. Since the contributions of the terms (55a) and (55d) tend to zero uniformly in y, $|y| \leqslant r$, as $\lambda \to \infty$, so do the corresponding integrals. With the asymptotic behavior (62) and (65) (which are also uniform in y), we are left with

$$\left\{ \left(\sum_{j=2}^n \sigma_j \right) \rho(Q) + \int dq\, \sigma[\rho(qQ) - \rho(q)\rho(Q)] \right\}$$
$$\times\ \sigma_1 d(\hat{u}) \int_{B(0,r)} \rho(\lambda\hat{u} + y, \sigma_1)\, dy$$
$$= \beta^{-1}\lambda^\gamma \int_{B(\lambda\hat{u},r)} \nabla g_\sigma(x)\, dx \qquad \text{as} \quad \lambda \to \infty \tag{67}$$

Performing the integration on the right-hand side of (67), we get with (58b) again

$$\lambda^\gamma \int_{B(\lambda\hat{u},r)} \nabla g_\sigma(x)\, dx = \lambda^\gamma \int_{\partial B(0,r)} g_\sigma(\lambda\hat{u} + y)\, dy = |\partial B(0, r)| \times o(1)$$

Therefore the right-hand side of (67) tends to zero as $\lambda \to \infty$. Since $d(\hat{u}) \neq 0$ and the state has a nonzero local density by assumption (b2), we conclude that

$$\left(\sum_{j=2}^n \sigma_j \right) \rho(Q) + \int dq\, \sigma[\rho(qQ) - \rho(q)\rho(Q)] = 0$$
$$Q = (q_2, \ldots, q_n), \qquad n \geqslant 2$$

These are precisely the sum rules (49). ■

Corollary. The equilibrium states of one-dimensional Coulomb systems (one- and two-component plasmas) constructed in Ref. 5 satisfy the canonical sum rules.

Indeed the equilibrium states obtained in Ref. 5 are RE states with exponential clustering; all the assumptions of Proposition 8 are therefore satisfied. ■

Remark. For Coulomb systems, in any dimension we see that any periodic RE state will satisfy the *canonical sum rules* as soon as "screening" occurs, i.e., as soon as (c1)–(c3) are satisfied (with $\gamma = \nu - 1$).

The next proposition applies to infinite-range integrable potentials. The proposition states that with additional assumptions on the derivatives of the truncated functions the sum rules must hold if the decay of the truncated functions is faster than the decay of the potential.

Proposition 9. Assume that the force $F(x)$ is of the form $F(x) = -\nabla\phi(x)$ with ϕ integrable and that $F(x)$ satisfies conditions (a1) and (a2) of Proposition 8 with $\gamma > \nu + 1$. Then any equilibrium state satisfying the following conditions obeys the sum rules (49):

(b′1) $E_\rho(x) = 0$ and (b2) as in Proposition 8.

(c′1) For x_2 fixed

$$\nabla_1 \rho_T^{(2)}(x_1\sigma_1, x_2\sigma_2) = o(1/|x_1|^\gamma)$$

(c′2) For $n \geqslant 3$ and $x_3,\ldots, x_n$ fixed

$$\nabla_1 \rho_T^{(n)}(x_1\sigma_1,\ldots, x_n\sigma_n) = o(1/|x_1|^\gamma)$$

uniformly in x_2, and (c3) as in Proposition 8.

Remark. Since here the force is integrable, ρ is shape independent. Notice that the effective field will necessarily be zero if ρ is Euclidean-invariant.

Conditions (c′1) and (c′2) give an upper bound on the rate of decrease of the derivatives of the truncated functions. Conditions (c′1) and (c′2) are stronger than (c1) and (c2) in the sense that now fast oscillations in the clustering are not allowed.

For the proof we proceed exactly as in Proposition 8.

We set $x_1 = \lambda\hat{u} + y$ and examine the asymptotic behavior of the terms of Eqs. (55). Now (c′1) and (c′2) imply

$$\nabla g_\sigma(x) = o(1/|x|^\gamma), \qquad |x| \to \infty \tag{68}$$

and therefore

$$g_\sigma(x) = o(1/|x|^{\gamma-1}) \tag{69}$$

$$g_\sigma(x_1) = o(1/\lambda^{\gamma-1}) \tag{70}$$

uniformly in y, $|y| \leqslant r$.

The term (55a) is zero by assumption. The term (55b) converges again to the expression (62). Since Lemma 5 still holds true (see Appendix), (58c) converges to (65) and (55d) vanishes. Therefore we get (67), and with (68)

$$\lambda^{\nu} \int_{B(\lambda\hat{u},r)} dx\, \nabla g_{\sigma}(x) = |B(0, r)| \times o(1)$$

From this the sum rules follow as in Proposition 8. ■

6. CONSEQUENCES OF THE SUM RULES

The most important feature of a state that obeys the sum rules is that the fluctuations of the charge are not normal. Let Q_{Λ} be the observable charge of the *particles* in the region Λ

$$Q_{\Lambda} = \sum_{x_j \in \Lambda} \sigma_j \chi_{\Lambda}(x_j), \qquad \chi_{\Lambda}(x) = \begin{cases} 1, & x \in \Lambda \\ 0, & x \notin \Lambda \end{cases} \tag{71}$$

We have

$$\langle Q_{\Lambda} \rangle = \int_{\Lambda} dx_1 \sum_{\sigma_1} \sigma_1 \rho^{(1)}(x_1 \sigma_1) \tag{72}$$

$$\langle Q_{\Lambda}^2 \rangle = \int_{\Lambda} dx_1 \int_{\Lambda} dx_2 \sum_{\sigma_1 \sigma_2} \sigma_1 \sigma_2 \rho^{(2)}(x_1\sigma_1, x_2\sigma_2) + \int_{\Lambda} dx_1 \sum_{\sigma_1} \sigma_1^2 \rho^{(1)}(x_1\sigma_1) \tag{73}$$

The fluctuations have a normal behavior if $\langle [Q_{\Lambda} - \langle Q_{\Lambda} \rangle]^2 \rangle = \langle Q_{\Lambda}^2 \rangle - \langle Q_{\Lambda} \rangle^2$ is extensive with Λ as $\Lambda \to \mathbb{R}^{\nu}$. The next proposition shows that this is not the case as soon as the sum rules hold.

Proposition 10. Let ρ be a RE state invariant under a discrete subgroup $\mathscr{T}$ of the translation group and Λ a sequence of volumes defined as union of N unit cells Δ_n,

$$\Delta_n = \{x + a_n;\quad x \in \Delta_0, \quad a_n \in \mathscr{T}\}$$

If the first sum rule (1) holds, then

$$\operatorname*{Lim}_{\Lambda \to \mathbb{R}^{\nu}} \frac{1}{|\Lambda|} \langle [Q_{\Lambda} - \langle Q_{\Lambda} \rangle]^2 \rangle = 0$$

Proof. We have from (72) and (73)

$$\frac{1}{|\Lambda|} [\langle Q_{\Lambda}^2 \rangle - \langle Q_{\Lambda} \rangle^2] = \frac{1}{|\Lambda|} \int_{\Lambda} dx_1 \int_{\Lambda} dx_2 \sum_{\sigma_1 \sigma_2} \sigma_1 \sigma_2 \rho_T^{(2)}(x_1\sigma_1, x_2\sigma_2) + \frac{1}{|\Lambda|} \int_{\Lambda} dx_1 \sum_{\sigma_1} \sigma_1^2 \rho^{(1)}(x_1\sigma_1) \tag{74}$$

Introducing $f(x_1x_2) = \sum_{\sigma_1\sigma_2} \sigma_1\sigma_2\rho_T^{(2)}(x_1\sigma_1, x_2\sigma_2)$, we have

$$\begin{aligned}
&\frac{1}{|\Lambda|}\int_\Lambda dx_1 \int_\Lambda dx_2\, f(x_1, x_2)\\
&\quad = \frac{1}{N|\Delta_0|}\sum_{\Delta_n\subset\Lambda}\int_{\Delta n} dx_1 \int_{\mathbb{R}^\nu} dx_2\, f(x_1, x_2)\chi_\Lambda(x_2)\\
&\quad = \frac{1}{N|\Delta_0|}\sum_{a_n\in\Lambda}\int_{\Delta_0} dx_1 \int dx_2\, f(x_1 + a_n, x_2)\chi_\Lambda(x_2)\\
&\quad = \frac{1}{|\Delta_0|}\int_{\Delta_0} dx_1 \int dx_2\, f(x_1, x_2)\frac{1}{N}\sum_{a_n\in\Lambda}\chi_\Lambda(x_2 + a_n)
\end{aligned} \tag{75}$$

Where we have used the periodicity of the state, i.e., $f(x_1, x_2) = f(x_1 + a_n, x_2 + a_n)$ if $a_n \in \mathscr{T}$, to obtain (75).

Now

$$1 \geqslant \frac{1}{N}\sum_{a_n\in\Lambda}\chi_\Lambda(x + a_n) = 1 - \frac{1}{N}\sum_{\substack{a_n\in\Lambda\\ a_n\notin\Lambda - x}} 1$$

and for fixed x

$$\frac{1}{N}\sum_{a_n\in\Lambda/(\Lambda - x)} 1 \leqslant \frac{|\Lambda \triangle (\Lambda - x)|}{|\Lambda|} + o(1) \to 0 \qquad \text{as} \quad |\Lambda| \to \infty$$

Since $f(x_1, x_2)$ is integrable in x_2 it follows by dominated convergence that

$$\operatorname*{Lim}_{\Lambda\to\mathbb{R}^\nu} \frac{1}{|\Lambda|}\int_\Lambda dx_1 \int_\Lambda dx_2 f(x_1, x_2) = \frac{1}{|\Delta_0|}\int_{\Delta_0} dx_1 \int_{\mathbb{R}^\nu} dx_2 f(x_1, x_2) \tag{76}$$

Moreover,

$$\operatorname*{Lim}_{\Lambda\to\mathbb{R}^\nu} \frac{1}{|\Lambda|}\int_\Lambda dx_1\, \rho^{(1)}(x_1\sigma_1) = \frac{1}{|\Delta_0|}\int_{\Delta_0} dx_1\, \rho^{(1)}(x_1\sigma_1) \tag{77}$$

Combining (74), (76), and (77), we get

$$\begin{aligned}
\operatorname*{Lim}_{\Lambda\to\mathbb{R}^\nu} \frac{1}{|\Lambda|}[\langle Q_\Lambda{}^2\rangle - \langle Q_\Lambda\rangle^2] = \frac{1}{|\Delta_0|}\int_{\Delta_0} dx_1 \sum_{\sigma_1}\sigma_1[\sigma_1\rho^{(1)}(x_1\sigma_1)\\
+ \int_{\mathbb{R}^\nu} dx_2 \sum_{\sigma_2}\sigma_2\rho_T^{(2)}(x_1\sigma_1, x_2\sigma_2)]
\end{aligned} \tag{78}$$

But the right-hand side of (78) is zero by the first sum rule. ■

This result has several interesting implications:

1. Consider a *several-component Coulomb system* with $\rho_B = 0$. Then Q_Λ is the total electric charge in Λ. If "screening" occurs as is expected, Q_Λ

cannot have normal fluctuations. Other properties of the charge fluctuations and of their connections with the sum rules will be discussed in Ref. 10. It is shown in Ref. 10 that, under suitable assumptions on the clustering, the charge fluctuations $\langle Q_\Lambda{}^2\rangle - \langle Q_\Lambda\rangle^2$ are necessarily of the order of the surface $|\partial\Lambda|$ of $|\Lambda|$. Moreover, if $\hat{Q}_\Lambda = (Q_\Lambda - \langle Q_\Lambda\rangle)/|\partial\Lambda|^{1/2}$ is the normalized charge observable, the probability distribution for $\hat{Q}_\Lambda$ is again Gaussian as $|\Lambda| \to \infty$ in two or three space dimensions. In one dimension, $|\partial\Lambda| = 1$, and the limiting probability distribution of the charge is discrete and can be found explicitly.

2. In a one-component system with $\sigma = 1$ the observable Q_Λ coincides with the particle number N_Λ in Λ and

$$\chi = \lim_{\Lambda\to\mathbb{R}^\nu} \frac{1}{|\Lambda|}\langle[N_\Lambda - \langle N_\Lambda\rangle]^2\rangle$$

represents the "bulk compressibility". Therefore, if the sum rules are satisfied, $\chi = 0$.

(a) Let us then consider first the "jellium" ($\rho_B > 0$, Coulomb force). We know by the corollary of Proposition 8 that in one dimension the sum rules are satisfied and thus $\chi = 0$. In any dimension the bulk compressibility χ will also be zero for any periodic RE state as soon as screening occurs.

Heuristically this result can be understood in relation to the fact that the long-range force imposes local neutrality, i.e., the local density $\bar{\rho} = (1/|\Delta_0|)\int_{\Delta_0} dx\, \rho^{(1)}(x)$ is equal to ρ_B. Therefore, for a given ρ_B the system appears to be incompressible, and thus $\chi = 0$.

(b) In the case of a *one-component fluid* of particles interacting with an integrable potential (and $\rho_B = 0$), Propositions 8 and 9 provide lower bounds on the rate of decrease of the truncated correlation functions. Indeed if we know that the particle fluctuations are normal, i.e.,

$$\chi \neq 0$$

the assumptions of Propositions 8 and 9 cannot hold true. Consider for instance an equilibrium state of a system of particles interacting with a stable, regular pair potential $\phi(x)$, where $\phi(x)$ behaves as $1/|x|^{\gamma-1}$ ($\gamma - 1 > \nu$) as $|x| \to \infty$. The correlation functions obey the Kirkwood–Saltzbourg equation and we know that at sufficiently high temperature the compressibility is not zero.[3]

Moreover, by the equivalence of equilibrium equations,[1] the correlation functions are also solutions of the BBGKY hierarchy. Therefore, if the force is as in Proposition 8 [resp., Proposition 9], we conclude that (c1) and (c2) [resp., (c′1) and (c′2)] cannot hold for the two- and three-point functions. (Otherwise, the first sum rule would imply $\chi = 0$.)

Proposition 8 implies then that $\rho_T^{(2)}$ and $\rho_T^{(3)}$ have to decrease as the force, i.e., as $1/|x|^\gamma$ or slower as $|x| \to \infty$ (assuming $\rho_T^{(2)}$ and $\rho_T^{(3)}$ have the same asymptotic behavior). In this connection, we recall the result of Duneau and Souillard[13] which says that the truncated correlation functions have to decrease faster than $1/|x|^{(\gamma-1)/2}$.

If we have the further information that $|x|\, \nabla\rho_T = O(\rho_T)$ on the decrease of the derivatives of the truncated functions, we use Proposition 9 to deduce that ρ_T cannot decrease faster than the potential itself as $|x| \to \infty$; on the other hand, it has been shown by Groeneveld[11] that $\rho_T^{(2)}$ has to decrease as the potential or faster (in the domain of analyticity). Using this result, we conclude therefore that $\rho_T^{(2)}$ decreases at infinity exactly like the potential (in the domain of analyticity).

Let us recall that a similar result has been derived for ferromagnetic lattice systems at large magnetic field by Iagolnitzer and Souillard.[14]

3. The sum rules impose an upper bound on the possible values of the internal field. Consider a *two-component system* with $\sigma = \pm 1$ and $\rho_B = 0$ which is translation invariant and neutral. Then

$$\rho^{(1)}(x\sigma) = \rho, \qquad \sigma = \pm 1, \qquad c_\rho = 0$$

$$\rho^{(2)}(x_1\sigma_1, x_2\sigma_2) = \rho^{(2)}_{\sigma_1\sigma_2}(x_1 - x_2) = \rho^{(2)}_{\sigma_2\sigma_1}(x_2 - x_1)$$

The first BBGKY equation yields

$$E = (1/\rho) \int dx\, F(x)[\rho^{(2)}_{+-}(x) - \rho^{(2)}_{++}(x)] \tag{79}$$

Assume that we have

$$\rho^{(2)}_{+-}(x) \geqslant \rho^{(2)}_{++}(x) \tag{80}$$

meaning that, given a positive charge at the origin, it is more likely to find a negative charge at x than another positive charge. Relation (80) is true for the one-dimensional Coulomb gas[6] and it is likely to hold whenever screening occurs.

Relations (79) and (80) and the first sum rule imply

$$|E| \leqslant \sup_x |F(x)| \tag{81}$$

If the force is bounded, (81) provides an upper bound for the possible values of E which is independent of the state. (81) can be interpreted as follows. Consider that the state has been obtained as the thermodynamic limit of Gibbs states with some boundary conditions or in external fields. Then, because of the occurrence of screening, the internal field cannot exceed a fixed value, no matter how strong the applied external field or the effects of the boundary conditions. An example of such a situation can be found in the one-dimensional Coulomb system.[9]

To conclude this section, we shall now show that the canonical sum rules are always satisfied in the canonical ensemble at finite volume.

Consider indeed a finite system consisting of $n_1,\dots, n_N$ particles of charges $\sigma^{(1)},\dots, \sigma^{(N)}$ in a box Λ. We denote by $x_j^{(I)}$, $j = 1,\dots, n_I$, $I = 1,\dots, N$, the coordinates of the particles of charge σ^I. By definition, the correlation functions are given by

$$\begin{aligned}&\rho_\Lambda(x_1^{(1)} \cdots x_{k_1}^{(1)};\dots; x_1^{(N)} \cdots x_{k_N}^{(N)})\\ &\quad= \prod_{I=1}^N \frac{n_I!}{(n_I - k_I)!} \prod_{I=1}^N \int_\Lambda dx_{k_I+1}^{(I)} \cdots dx_{n_I}^{(I)}\, \rho_\Lambda(x_1^{(1)} \cdots x_{n_1}^{(1)};\dots; x_1^{(N)} \cdots x_{n_N}^{(N)})\end{aligned}$$

and we have

$$\begin{aligned}&\sum_{I=1}^N \sigma^I \int_\Lambda dx_{k_I+1}^{(I)}\, [\rho_\Lambda(\dots; x_1^{(I)} \cdots x_{k_I+1}^{(I)};\dots) - \rho_\Lambda(x_{k_I+1}^{(I)})\rho_\Lambda(\dots; x_1^{(I)},\dots, x_{k_I}^{(I)};\dots)]\\ &\quad= \sum_{I=1}^N \sigma^{(I)}[(n_I - k_I)\rho_\Lambda(\dots; x_1^{(I)},\dots, x_{k_I}^{(I)};\dots) - n_I\rho_\Lambda(\dots; x_1^{(I)},\dots, x_{k_I}^{(I)};\dots)]\\ &\quad= -\sum_{I=1}^N \sigma^{(I)} k_I \rho_\Lambda(\dots;\dots;\dots)\end{aligned}$$

i.e.,

$$\begin{aligned}&\sum_\sigma \sigma \int_\Lambda dx\, [\rho_\Lambda^{(k+1)}(x_1\sigma_1,\dots, x_k\sigma_k, x\sigma) - \rho_\Lambda^{(1)}(x\sigma)\rho_\Lambda^{(k)}(x_1\sigma_1,\dots, x_k\sigma_k)]\\ &\quad= -\left(\sum_{i=1}^N \sigma_i\right)\rho_\Lambda^{(k)}(x_1\sigma_1,\dots, x_k\sigma_k)\end{aligned}$$

which are precisely the sum rules.

Although the sum rules hold trivially for finite systems irrespective of the nature of the forces, the same sum rules will not be true in general in infinite systems. For instance, if we take the thermodynamic limit of a system of noninteracting particles, we get

$$\lim_{\Lambda \to \mathbb{R}^\nu} \rho_\Lambda^{(1)}(x, \sigma) = \rho_\sigma^{(1)}$$

$$\lim_{\Lambda \to \mathbb{R}^\nu} \rho_\Lambda^{(k)}(x_1\sigma_1,\dots, x_k\sigma_k) = \prod_{i=1}^k \rho_{\sigma_i}^{(1)}$$

Obviously these correlation functions do not satisfy the sum rules.

More generally, the correlation functions of an infinite system of a single kind of particle interacting with short-range forces are not expected to satisfy the sum rules, on the physical grounds that the compressibility should not vanish, as discussed at the beginning of the section.

On the other hand, it is also clear that when the force has finite range and the truncated correlation functions have infinite range (say exponential decay), Propositions 8 and 9 are not applicable.

However, the sum rules are an essential characteristic of a system with long-range forces. In particular, any approximation scheme for the treatment of Coulomb systems based on the BBGKY hierarchy should be consistent with the sum rules. Such approximations have already been used in the literature under the name "convolution approximation."[14]

APPENDIX

Proof of Lemma 1

(i) We notice first that when the sequences (V_λ) satisfy condition (b), $(V_\lambda{}^1)\ \alpha_1\ (V_\lambda{}^2)$ if and only if there exist c' and c'' such that

$$c' \leqslant R_{2,\lambda}^-/R_{1,\lambda}^- \leqslant c'', \qquad 0 < c', c'' < \infty \tag{A1}$$

$R_{j,\lambda}^-$ are the radii of the internal balls associated with $(V_\lambda{}^j)$.

Indeed, by (b) we have $1 \leqslant R_{2,\lambda}^+/R_{2,\lambda}^- \leqslant \delta_2$, $1 \leqslant R_{1,\lambda}^+/R_{1,\lambda}^- \leqslant \delta_1$. Since $V_\lambda{}^1 \triangle V_\lambda{}^2 \subset B_{1,\lambda}^+ \cup B_{2,\lambda}^+$ and $V_\lambda{}^1 \cap V_\lambda{}^2 \supset B_{1,\lambda}^- \cap B_{2,\lambda}^-$, we get

$$\frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|} \leqslant \left(\frac{\sup(R_{1,\lambda}^+, R_{2,\lambda}^+)}{\inf(R_{1,\lambda}^-, R_{2,\lambda}^-)}\right)^\nu \leqslant (\sup\{\delta_1, \delta_2, \delta_2 c'', \delta_1/c'\})^\nu$$

showing that (A1) implies the relation α_1.

Conversely, assume first that $R_{2,\lambda}^-/R_{1,\lambda}^- \geqslant 1$; then, either $R_{2,\lambda}^- \leqslant R_{1,\lambda}^+$, giving $R_{2,\lambda}^-/R_{1,\lambda}^- \leqslant R_{1,\lambda}^+/R_{1,\lambda}^- \leqslant \delta_1$, or $R_{2,\lambda}^- > R_{1,\lambda}^+$, which gives

$$\frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|} \geqslant \frac{(R_{2,\lambda}^-)^\nu - (R_{1,\lambda}^+)^\nu}{(R_{1,\lambda}^+)^\nu} \geqslant \left(\frac{R_{2,\lambda}^-}{R_{1,\lambda}^-}\right)^\nu \frac{1}{\delta_1{}^\nu} - 1$$

Therefore,

$$1 \leqslant \frac{R_{2,\lambda}^-}{R_{1,\lambda}^-} \leqslant \left(1 + \frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|}\right)^{1/\nu} \delta_1$$

Inverting the roles of 1 and 2, we obtain in all cases the inequality

$$\left[\left(1 + \frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|}\right)^{1/\nu} \delta_2\right]^{-1} \leqslant \frac{R_{2,\lambda}^-}{R_{1,\lambda}^-} \leqslant \delta_1 \left(1 + \frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|}\right)^{1/\nu}$$

and thus the relation α_1 implies the inequality (A1).

(ii) The relation α_1 is an equivalence relation; indeed it is sufficient to verify the transitivity. Writing $R_{3,\lambda}^-/R_{1,\lambda}^- = (R_{3,\lambda}^-/R_{2,\lambda}^-) R_{2,\lambda}^-/R_{1,\lambda}^-$, the transitivity follows obviously from (A1).

(iii) The relation α_2 is an equivalence relation; again it is sufficient to prove the transitivity. Notice that

$$|V_\lambda{}^1 \triangle V_\lambda{}^3| \leqslant |V_\lambda{}^1 \triangle V_\lambda{}^2| + |V_\lambda{}^2 \triangle V_\lambda{}^3|$$

from which it follows that

$$\frac{|V_\lambda{}^1 \triangle V_\lambda{}^3|}{|V_\lambda{}^1 \cap V_\lambda{}^3|} \leqslant \frac{|V_\lambda{}^1 \triangle V_\lambda{}^2|}{|V_\lambda{}^1 \cap V_\lambda{}^2|} + \frac{|V_\lambda{}^2 \triangle V_\lambda{}^3|}{|V_\lambda{}^2 \cap V_\lambda{}^3|}$$

This inequality concludes the proof, since $\lim_{\lambda\to\infty}(|V_\lambda^1 \triangle V_\lambda^2|/|V_\lambda^1 \cup V_\lambda^3|) = 0$ implies

$$\lim_{\lambda\to\infty} \frac{|V_\lambda^1 \triangle V_\lambda^3|}{|V_\lambda^1 \cap V_\lambda^3|} = \lim_{\lambda\to\infty} \frac{|V_\lambda^1 \triangle V_\lambda^3|}{|V_\lambda^1 \cup V_\lambda^3| - |V_\lambda^1 \triangle V_\lambda^3|} = 0$$

(iv) The relation α_3 is trivially an equivalence relation.

(v) Finally, we show that if the sequences of volumes satisfy the condition (a) [van Hove], then

$$\alpha_3 \Rightarrow \alpha_2 \Rightarrow \alpha_1$$

If $V_\lambda^2 = V_\lambda^1 + a$ with $a \in \mathbb{R}^\nu$, then for λ large enough so that $|V_\lambda^{|a|}| < |V_\lambda|$, we have

$$\frac{|V_\lambda \triangle (V_\lambda + a)|}{|V_\lambda \cap (V_\lambda + a)|} \qquad \frac{2|V_\lambda^{|a|}|}{|V_\lambda| - |V_\lambda^{|a|}|}$$

Since $V_\lambda \to \mathbb{R}^\nu$ in the sense of van Hove, we have

$$\lim_{\lambda\to\infty} \frac{|V_\lambda \triangle (V_\lambda + a)|}{|V_\lambda \cap (V_\lambda + a)|} = 0$$

i.e., $\alpha_3 \Rightarrow \alpha_2$.

$\alpha_2 \Rightarrow \alpha_1$ is obvious. ■

Proof of the Equivalence Between (49) and (50)

(i) Let us show that (49) implies (50). By definition of the truncated functions,

$$\rho_T^{(n)}(Q) = \sum_{\mathscr{Q}} (-1)^{k-1}(k-1)! \prod_{\alpha=1}^{k} \rho(Q_\alpha)$$

where $\sum_{\mathscr{Q}} \equiv \sum_{Q=\cup_{\alpha=1}^k Q_\alpha}$ means "summation over all partitions of Q into k disjoint subsets, $k = 1, 2,\dots, n$." We thus have, using (49),

$$\begin{aligned}
&\left(\sum_{j=1}^{n} \sigma_j\right) \rho_T^{(n)}(Q) \\
&= \sum_{\mathscr{Q}} (-1)^{k-1}(k-1)! \sum_{\alpha=1}^{k} \left[\prod_{\beta\neq\alpha} \rho(Q_\beta)\right]\left[\left(\sum_{q_j\in Q_\alpha} \sigma_j\right)\rho(Q_\alpha)\right] \\
&= -\sum_{\mathscr{Q}} (-1)^{k-1}(k-1)! \sum_{\alpha=1}^{k} \prod_{\beta\neq\alpha} \rho(Q_\beta)\left\{\int d\bar{q}\, \bar{\sigma}[\rho(Q_\alpha\bar{q}) - \rho(Q_\alpha)\rho(\bar{q})]\right\} \\
&= -\int d\bar{q}\, \bar{\sigma}\left[\sum_{\mathscr{Q}} (-1)^{k-1}(k-1)! \sum_{\alpha=1}^{k} \prod_{\beta\neq\alpha} \rho(Q_\beta)\rho(Q_\alpha\bar{q})\right. \\
&\qquad \left. + \sum_{\mathscr{Q}} (-1)^k k! \prod_{\alpha=1}^{k} \rho(Q_\alpha)\rho(\bar{q})\right] \\
&= -\int d\bar{q}\, \bar{\sigma}\rho_T^{(n+1)}(Q\bar{q})
\end{aligned}$$

which yields (50).

(ii) Let us then show that (50) implies (49). By definition of the truncated functions,

$$\rho(Q) = \sum_{\mathscr{Q}} \prod_{\alpha=1}^{k} \rho_T(Q_\alpha) \tag{A2}$$

We thus have, using (50),

$$\begin{aligned}
\left(\sum \sigma_j\right)\rho(Q) &= \sum_{\mathscr{Q}} \sum_{\alpha=1}^{k} \left[\prod_{\beta\neq\alpha} \rho_T(Q_\beta)\right]\left[\left(\sum_{q_j=Q_\alpha} \sigma_j\right)\rho_T(Q_\alpha)\right] \\
&= -\sum_{\mathscr{Q}} \sum_{\alpha=1}^{k} \left[\prod_{\beta\neq\alpha} \rho_T(Q_\beta)\right]\int d\bar{q}\,\bar{\sigma}\rho_T(Q_\alpha\bar{q}) \\
&= -\int d\bar{q}\,\bar{\sigma}\left[\sum_{\mathscr{Q}} \sum_{\alpha=1}^{k} \prod_{\beta\neq\alpha} \rho_T(Q_\beta)\rho_T(Q_\alpha\bar{q})\right] \\
&= -\int d\bar{q}\,\bar{\sigma}[\rho(Q\bar{q}) - \rho(Q)\rho(\bar{q})]
\end{aligned}$$

which yields (49). ∎

Proof of Lemma 4

Using the definition (A2), we have

$$\rho(qQ) - \rho(Q)\rho(q) = \sum_{\phi\neq Q_1\subset Q} \rho_T(qQ_1)\rho(Q\backslash Q_1) \tag{A3}$$

and

$$\begin{aligned}
&\rho(qq_1Q) - \rho(q_1Q)\rho(q) \\
&\quad= \sum_{\{q_1Q\}=\cup Q_\alpha} \rho_T(qQ_1)\prod_{\alpha\neq 1} \rho_T(Q_\alpha) \\
&\quad= \sum_{Q=\cup Q_\alpha} \rho_T(qq_1)\prod_{\alpha} \rho_T(Q_\alpha) + \sum_{Q=\cup Q_\alpha} \rho_T(qq_1Q_1)\prod_{\alpha\neq 1} \rho_T(Q_\alpha) \\
&\qquad+ \sum_{Q=\cup Q_\alpha} \rho_T(qQ_1)\left[\sum_{\beta\neq 1} \rho_T(Q_\beta q_1)\prod_{\substack{\alpha\neq\beta\\ \alpha\neq 1}} \rho_T(Q_\alpha) + \rho(q_1)\prod_{\alpha\neq 1} \rho_T(Q_\alpha)\right] \\
&\quad= \rho_T(qq_1)\rho(Q) + \sum_{\phi\neq Q_1\subset Q} \rho_T(qq_1Q_1)\rho(Q\backslash Q_1) \\
&\qquad+ \sum_{\phi\neq Q_1\subset Q} \rho_T(qQ_1)\rho(q_1Q\backslash Q_1) \\
&\quad= \rho_T(qq_1)\rho(Q) + \sum_{\phi\neq Q_1\subset Q} \{\rho_T(qq_1Q_1)\rho(Q\backslash Q_1) + \rho_T(qQ_1)\rho(q_1Q\backslash Q_1)\} \\
&\qquad- \sum_{\phi\neq Q_1\subset Q} \rho_T(qQ_1)[\rho(Q\backslash Q_1)\rho(q_1) - \rho(Q\backslash Q_1)\rho(q_1)]
\end{aligned}$$

Using (A3),

$$
\begin{aligned}
&= \rho_T(qq_1)\rho(Q) + \rho(q_1)[\rho(qQ) - \rho(q)\rho(Q)] \\
&\quad + \sum_{\phi \neq Q_1 \subset Q} \{\rho_T(qQ_1)[\rho(q_1 Q\backslash Q_1) - \rho(q_1)\rho(Q\backslash Q_1)] \\
&\quad + \rho_T(qq_1 Q_1)\rho(Q\backslash Q_1)\}
\end{aligned}
$$

which concludes the proof, using (A3) once more for the expression

$$[\rho(q_1 Q\backslash Q_1) - \rho(q_1)\rho(Q\backslash Q_1)]$$

Proof of Lemma 5

Consider first the case $\gamma > \nu$. Choose $\epsilon > 0$ and λ large enough so that by (58) and (a1)

$$|g_\sigma(x - y)| \leqslant \epsilon/|x|^\gamma, \qquad |x| > \lambda/2, \quad |y| \leqslant r \tag{A4}$$

$$|F(x)| \leqslant M_2/|x|^\gamma, \qquad |x| > \lambda/2 \tag{A5}$$

Then, we decompose $\mathbb{R}^\nu = \Sigma_1 \cup \Sigma_2$, where Σ_1 is the ball $\{x; |x| \leqslant \lambda/2\}$ and $\Sigma_2 = \{x; |x| > \lambda/2\}$, and write

$$
\begin{aligned}
I(\lambda) &= \lambda^\gamma \int_{\mathbb{R}^\nu} F(\lambda\hat{u} + y - x) g_\sigma(x)\, dx = \lambda^\gamma \int_{\mathbb{R}^\nu} F(\lambda\hat{u} - x) g_\sigma(x - y)\, dx \\
&= I_1(\lambda) + I_2(\lambda)
\end{aligned} \tag{A6}
$$

where $I_1(\lambda)$ and $I_2(\lambda)$ are the integrals over Σ_1 and Σ_2, respectively.

In Σ_1, $|\lambda\hat{u} - x| \geqslant |\lambda - |x|\,| \geqslant \lambda/2$; thus by (A5)

$$|\lambda^\gamma F(\lambda\hat{u} - x) g_\sigma(x - y)| \leqslant 2^\gamma M_2 |g_\sigma(x - y)|$$

Since $g_\sigma \in \mathscr{L}^1$, we can apply the dominated convergence theorem, which yields, taking account of (60),

$$\lim_{\lambda \to \infty} I_1(\lambda) = d(\hat{u}) \int_{\mathbb{R}^\nu} g_\sigma(x)\, dx \tag{A7}$$

In Σ_2, $|x| > \lambda/2$ and (A4) gives

$$I_2(\lambda) \leqslant \epsilon \int_{\Sigma_2} |F(\lambda\hat{u} - x)|\, dx \leqslant \epsilon \|F\|_1 \tag{A8}$$

(A7) and (A8) give the desired result.

We now treat the case $\gamma \leqslant \nu$.

Choose a number μ such that

$$|F(x)| \leqslant M_2/|x|^\gamma, \qquad |x| \geqslant \mu \tag{A9}$$

As above, take λ such that $\lambda/2 > \mu$ and

$$|g_\sigma(x - y)| \leqslant M_1/|x|^{\nu+\epsilon}, \qquad |x| \geqslant \lambda/2, \quad |y| \geqslant r \tag{A10}$$
$$|F(x)| \leqslant M_2/|x|^{\gamma}, \qquad |x| \geqslant \lambda/2$$

We decompose $\mathbb{R}^\nu$ into four regions:

$$\Sigma_1 = \{x; |x| \leqslant \lambda/2\}$$
$$\Sigma_2 = \{x: |x| \geqslant 3\lambda/2\}$$
$$\Sigma_3 = \{x; |\lambda\hat{u} - x| \leqslant \mu\}$$
$$\Sigma_4 = \{x; \lambda/2 < |x| < 3\lambda/2, \text{ and } x \notin \Sigma_3\}$$

Σ_4 is the region included between the two regions Σ_1 and Σ_2 lessened by the ball Σ_3 centered in $\lambda\hat{u}$ of fixed radius μ.

We write

$$I(\lambda) = I_1(\lambda) + I_2(\lambda) + I_3(\lambda) + I_4(\lambda) \tag{A11}$$

We have exactly, as for (A7),

$$\lim_{\lambda\to\infty} I_1(\lambda) = d(\hat{u}) \int_{\mathbb{R}^\nu} g_\sigma(x)\, dx \tag{A12}$$

We show that the other contributions to $I(\lambda)$ vanish as $\lambda \to \infty$. In Σ_2, $|\lambda\hat{u} - x| \geqslant |\,|x| - \lambda| > \lambda/2$ and (A5) applies:

$$I_1(\lambda) \leqslant M_2 \int_{\Sigma_2} |g_\sigma(x - y)|\, dx \to 0, \qquad \lambda \to \infty$$

since g_σ is integrable. In Σ_3, $|x| > \lambda/2$; then, by (A10),

$$I_3(\lambda) \leqslant \frac{M_1}{\lambda^{\nu-\gamma+\epsilon}} \int_{\Sigma_3} |F(\lambda\hat{u} - x)|\, dx = \frac{M_1}{\lambda^{\nu-\gamma+\epsilon}} \int_{|x|\leq\mu} |F(x)|\, dx$$

Since $F(x)$ is locally integrable and μ is fixed, this quantity tends obviously to zero as $\lambda \to \infty$. In Σ_4, both (A9) and (A10) apply; thus

$$I_4(\lambda) \leqslant \frac{M_1 M_2}{\lambda^{\nu-\gamma+\epsilon}} \int_{\Sigma_4} \frac{dx}{|\lambda\hat{u} - x|^{\gamma}} \tag{A13}$$

An estimate of potential theory[16] states that for $\gamma \leqslant \nu$

$$\int_D \frac{d^\nu x}{|a - x|^\gamma} = \begin{cases} O(r^{\nu-\gamma}), & \gamma < \nu \\ O(\lg r), & \gamma = \nu \end{cases}$$

where D is some finite region (not including a when $\gamma = \nu$) and r is the radius of a ball of volume equal to that of D. Since the volume of Σ_4 is proportional to λ^ν as $\lambda \to \infty$, we get

$$\int_{\Sigma_4} \frac{d^\nu x}{|\lambda\hat{u} - x|^\gamma} = \begin{cases} O(\lambda^{\nu-\gamma}), & \gamma < \nu \\ O(\lg \lambda), & \gamma = \nu \end{cases}$$

and this, combined with (A13), concludes the proof of the lemma. ■

For the proof of (66) we proceed exactly as in Lemma 5 with $R(x\sigma, x_1\sigma_1, Q)$ replacing $g_\sigma(x)$, and define the corresponding integrals $I_j(\lambda)$, $j = 1, 2, 3, 4$. The difference is that now $R(x\sigma, x_1\sigma_1, Q)$, $x_1 = \lambda\hat{u} + y$, depends itself on λ. But since we assume that the behavior (c2) of the truncated correlation functions is uniform with respect to x_1, we still have the equivalent of the estimates (A4) and (A10), i.e.,

$$|R(x\sigma, x_1\sigma_1, Q)| \leqslant \begin{cases} \epsilon/|x|^\gamma, & \gamma > \nu \\ M_1/|x|^{\nu+\epsilon}, & \gamma \leqslant \nu \end{cases} \quad \text{for } |x| \geqslant \lambda/2; \quad |y| \leqslant r \tag{A14}$$

Therefore, the proof that the integrals $I_2(\lambda)$, $I_3(\lambda)$, and $I_4(\lambda)$ vanish as $\lambda \to \infty$, which is based on (A14), requires no modification. Thus we get

$$\lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty} I_1(\lambda) \tag{A15}$$

But if x is in Σ_1, then $|\lambda^\gamma F(\lambda\hat{u} - x)| \leqslant 2^\gamma M_2$ [see (A7)] and thus

$$\begin{aligned} I_1(\lambda) &= \lambda^\gamma \int_{\Sigma_1} F(\lambda\hat{u} - x) R(x - y\sigma, x_1\sigma_1, Q)\, dx \\ &\leqslant 2^\gamma M_2 \int_{\mathbb{R}^\nu} |R(x\sigma, x_1\sigma_1, Q)|\, dx \end{aligned} \tag{A16}$$

The conjunction of (A15) and (A16) establishes (66). ■

For the proof of Proposition 8, Lemma 5 is modified as follows. Under the assumptions (a1), (68), and (69) we can choose $\epsilon > 0$ and λ such that

$$\left.\begin{aligned} |\nabla g_\sigma(x - y)| &\leqslant \epsilon/|x|^\gamma \quad &\text{(A17)} \\ |g_\sigma(x - y)| &\leqslant \epsilon/|x|^{\gamma-1} \quad &\text{(A18)} \end{aligned}\right\} \quad \text{for } |x| \geqslant \lambda/2, \quad |y| \leqslant r$$

and

$$\left.\begin{aligned} |F(x)| &\leqslant M_2/|x|^\gamma \quad &\text{(A19)} \\ |\phi(x)| &\leqslant M_2/|x|^{\gamma-1} \quad &\text{(A20)} \end{aligned}\right\} \quad \text{for } |x| \geqslant \lambda/2$$

Write $I(\lambda) = I_1(\lambda) + I_2(\lambda)$ as in (A6); then

$$\lim_{\lambda\to\infty} I_1(\lambda) = d(\hat{u}) \int_{\mathbb{R}^\nu} g_\sigma(x)\, dx$$

as in (A7). After an integration by parts, $I_2(\lambda)$ reads

$$I_2(\lambda) = \lambda^\gamma \int_{|x|>\lambda/2} \nabla_x \phi(\lambda\hat{u} - x) g_\sigma(x - y)\, dx$$

$$= \lambda^\gamma \int_{\partial B(0,\lambda/2)} \phi(\lambda\hat{u} - x) g_\sigma(x - y)\, ds$$

$$- \lambda^\gamma \int_{|x|>\lambda/2} \phi(\lambda\hat{u} - x)\, \nabla_x g_\sigma(x - y)\, dx \qquad \text{(A21)}$$

In the first integral of (A21), $|x| = \lambda/2$, and by (A18) and (A20) this integral is majorized by

$$\left|\partial B\left(0, \frac{\lambda}{2}\right)\right| \lambda^\gamma \frac{2^{\gamma-1} M_2}{\lambda^{\gamma-1}} \frac{2^{\gamma-1}\epsilon}{\lambda^{\gamma-1}} \leqslant \text{cst} \frac{\epsilon}{\lambda^{\gamma-\nu-1}}, \qquad \gamma > \nu + 1$$

With (A17) the second integral of (A21) is less than

$$2^\gamma \epsilon \int_{|x|>\lambda/2} |\phi(\lambda\hat{u} - x)|\, dx \leqslant 2^\gamma \epsilon \|\phi\|_1$$

This shows that $\lim_{\lambda\to\infty} I_2(\lambda) = 0$ and concludes the proof of the lemma. ■

The estimate (66) is established exactly as above.

NOTE ADDED IN PROOF

The validity of the sum rules has been noticed by Dyson [17] and Mehta and Dyson[18] in their analysis of the logarithmic potential occurring in the theory of random matrices (see the remark (c) in section V of Ref. 18).

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